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Spectral method for fractional quadratic Riccati differential equation

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Abstract

Fractional differentials provide more accurate models of systems under consideration. In this paper, approximation techniques based on the shifted Legendre spectral method is presented to solve fractional Riccati differential equations. The fractional derivatives are described in the Caputo sense. The technique is derived by expanding the required approximate solution as the elements of shifted Legendre polynomials. Using the operational matrix of the fractional derivative the problem can be reduced to a set of nonlinear algebraic equations. From the computational point of view, the solution obtained by this method is in excellent agreement with those obtained by previous work in the literature and also it is efficient to use.

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1 Introduction

Ordinary and partial fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [1]. Consequently, considerable attention has been given to the solutions of fractional differential equations of physical interest. Most fractional differential equations do not have exact solutions, so approximation and numerical techniques [2],[3],[4],[5], must be used. Recently, several numerical methods to solve the fractional differential equations have been given such as variational iteration method[6], homotopy perturbation method[8], Adomian's decomposition method [7], homotopy analysis method [9] and collocation method[10]. We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1. Caputo's definition of the fractional-order derivative is defined as [1]

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt, \qquad n-1 < \alpha \le n, n \in N,$$

where α is the order of the derivative and n is the smallest integer greater than α. For the Caputo's derivative we have:

$$D^{\alpha} C = 0, \qquad C \textit{ is a constant},$$

$$D^{\alpha} x^{\beta} = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \alpha \rceil \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in N_0 \text{ and } \beta \ge \lceil \alpha \rceil \end{cases}$$

We use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α. Also $N = 1,2,...$ and $N_0 = 0,1,2,...$. Recall that for $\alpha \in N$, the Caputo differential operator coincides with the usual differential operator of integer order.The main goal in this article is concerned with the application of Legendre spectral method to obtain the numerical solution of fractional Riccati differential equation [11], [12], [13].

$$D^{\alpha}u(x) = a(x) + b(x)u + g(x)u^2, \quad 0 \le x \le 1, \qquad m-1 < \alpha \le m, \tag{1}$$

with initial conditions

$$u^{(i)}(0) = d_i, \quad i = 0,1,...m-1, \tag{2}$$

where the fractional differential operator D^{α} is defined as in definition 1 and where a(x), b(x) and g(x) are given functions d_i, i=0,1,...m-1, are arbitrary constants and α, is a parameter describing the order of the fractional derivative.

The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of α, the fractional equation reduces to the classical Riccati differential equation. In the present paper we intend to extend the application of Legendre polynomials to solve fractional differential equations. Our main aim is to generalize Legendre operational matrix to fractional calculus.

The organization of this paper is as follows. In the next section we describe the basic formulation of shifted Legendre polynomials. Section 3 summarizes the application of Legendre spectral method to solve Eqs. (1, 2). As a result, a system of nonlinear ordinary differential equations is formed and the solution of the considered problem is introduced. In Section 4, some comparisons and numerical results are given to clarify the method. Figures and Tables are presented in section 5. And also, a conclusion is given in Section 6.

2 Shifted Legendre polynomials

The well-known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulas:

$$p_0(z) = 1, \qquad p_1(z) = z,$$

$$p_{i+1}(z) = \frac{2i+1}{i+1} z p_i(z) - \frac{i}{i+1} p_{i-1}(z), \quad i = 1,2,....$$

In order to use these polynomials on the interval[0,1], we define the so called shifted Legendre polynomials by introducing the change of variable

$$z = 2x - 1, \qquad 0 \le x \le 1.$$

The shifted Legendre polynomials in x are then obtained as follows:

$$p_0(x) = 1, \qquad p_1(x) = 2x - 1,$$

$$p_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1} p_i(x) - \frac{i}{i+1} p_{i-1}(x), \quad i = 1,2,....$$

The analytic form of the shifted Legendre polynomial $p_i(x)$ of degree i given by

$$p_i(x) = \sum_{k=0}^{i} (-1)^{(i+k)} \frac{(i+k)!\, x^k}{(i-k)!\, k!^2}. \tag{3}$$

Note that $p_i(0) = (-1)^i$ and $p_i(1) = 1$. The orthogonality condition is

$$\int_0^1 p_i(x) p_j(x) dx = \begin{cases} 0 & \text{for } i = j \\ \dfrac{1}{2i+1} & \text{for } i \neq j. \end{cases}$$

A function $y(x)$, square integrable in $[0,1]$, may be expressed in terms of the shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j p_j(x),$$

where the coefficients c_j are given by

$$c_j = (2j+1) \int_0^1 y(x) p_j(x) dx, \quad j = 1,2,....$$

In practice, only the first (m + 1)-terms shifted Legendre polynomials are considered. Then we have

$$y_m(x) = \sum_{j=0}^{m} c_j p_j(x) = C^T \phi(x), \tag{4}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\phi(x)$ are given by

$$C^T = [c_0, ..., c_m], \quad \phi(x) = [p_0(x), p_1(x), ..., p_m(x)]^T. \tag{5}$$

The derivative of the vector $\phi(x)$ can be expressed by

$$\frac{d\phi}{dx} = D^{(1)}\phi(x),$$

where $D^{(1)}$ is the $(m+1)(m+1)$ operational matrix of derivative and for odd m given as

$$D = 2\begin{pmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & ... & 0 & 0 & 0 \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ 0 & 3 & 0 & 7 & ... & 2m-3 & 0 & 0 \\ 1 & 0 & 5 & 0 & ... & 0 & 2m-1 & 0 \end{pmatrix}$$

and for even m given as

$$D = 2\begin{pmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & ... & 0 & 0 & 0 \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ . & . & . & . & ... & . & . & . \\ 1 & 0 & 5 & 0 & ... & 2m-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & ... & 0 & 2m-1 & 0 \end{pmatrix}$$

It is clear that

$$\frac{d^n\phi}{dx^n} = (D^{(1)})^n \phi(x),$$

where $n \in N$ and the superscript, in D^1 denotes matrix powers. Then

$$D^n = (D^{(1)})^n \qquad n = 1,2,.... \tag{6}$$

Theorem 1. Let $\phi(x)$ be the shifted Legendre vector defined in (5), and also suppose $\alpha > 0$ then

$$D^\alpha \phi(x) \approx D^{(\alpha)} \phi(x), \tag{7}$$

where $D^{(\alpha)}$ is the $(m+1)(m+1)$ operational matrix of fractional derivative of order α in Caputo sense and is defined as follows:

$$\begin{pmatrix}
0 & 0 & \cdots & 0 \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
0 & 0 & \cdots & 0 \\
\sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \Theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \Theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil} \Theta_{\lceil\alpha\rceil,m,k} \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\sum_{k=\lceil\alpha\rceil}^{i} \Theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i} \Theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i} \Theta_{i,m,k} \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
\sum_{k=\lceil\alpha\rceil}^{m} \Theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m} \Theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m} \Theta_{m,m,k}
\end{pmatrix}$$

Where $\Theta_{i,j,k}$ is given by

$$\Theta_{i,j,k} = 2j+1\sum_{l=0}^{j}\frac{(-1)^{(i+j+k+l)}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k-\alpha+1)(j-l)!(l!)^2(k+l+1-\alpha)}.$$

Proof. The proof is in [16]. □

Note that in $D^{(\alpha)}$, the first $\lceil\alpha\rceil$ rows, are all zero and if $\alpha = n \in N$, then Theorem 1 gives the same result as (6).

3 Applications of the operational matrix of fractional derivative

In this section, we Consider the Eqs. (1, 2). In order to use Legendre collocation method, we first approximate $u(x)$ as

$$u(x) \approx \sum_{i=0}^{m} c_i p_i(x) = C^T\phi(x) \tag{8}$$

where vector $C = [c_0,...,c_m]$ is an unknown vector. By using operational matrix of fractional derivative we have:

$$C^T D^{(\alpha)}\phi(x) - a(x) - b(x)C^T\phi(x) - g(x)(C^T\phi(x))^2 \tag{9}$$

we now collocate Eq. (9) at $(m+1-\lceil\alpha\rceil)$ points x_p as:

$$C^T D^{(\alpha)}\phi(x_p) - a(x_p) - b(x_p)C^T\phi(x_p) - g(x_p)(C^T\phi(x_p))^2, \quad p = 0,1,...,m-\lceil\alpha\rceil. \tag{10}$$

For suitable collocation points we use roots of shifted Legendre $p_{m+1-\lceil\alpha\rceil}(x)$. Eq. (10), together with $\lceil\alpha\rceil$ equations of the boundary conditions, give (m + 1) equations which can be solved, for the unknown u_i, $i = 0,...,m$.

4 Numerical results

In this section, we illustrate efficiency and accuracy of the presented method by the following numerical examples.

Example 1. Consider the following fractional Riccati equation:

$$\frac{d^{\alpha}u}{dt^{\alpha}} = -u^2(t)+1, \quad 0<\alpha\leq 1 \tag{11}$$

subject to the initial condition

$$u(0) = 0. \tag{12}$$

The exact solution, when $\alpha=1$, is

$$u(t) = \frac{e^{2t}-1}{e^{2t}+1}, \tag{13}$$

and we can observe that,as $t\rightarrow\infty$, $u(t)\rightarrow 1$. The obtained numerical results by means of the proposed method are shown in Table 1 and Figure 1.

In Table 1, comparison between the exact solution, the numerical solution using [12] and the approximate solution using our proposed method for $\alpha=1$ are presented. Note that as α approaches 1, the numerical solution converges to the analytical solution i.e. in the limit, the solution of the fractional differential equations approaches to that of the integer-order differential equations that it is shown in Figure 1.

Example 2. Consider the following fractional Riccati equation:

$$\frac{d^{\alpha}u}{dt^{\alpha}} = 2u(t)-u^2(t)+1, \quad 0<\alpha\leq 1, \tag{14}$$

subject to the initial condition

$$u(0) = 0. \tag{15}$$

The exact solution,when $\alpha=1$,is

$$u(t) = 1 + \sqrt(2)\tanh(\sqrt(2)x + \frac{1}{2}\log(\frac{\sqrt(2)-1}{\sqrt(2)+1})) \qquad (16)$$

and we can observe that, as $t \to \infty, u(t) \to 1 + \sqrt(2)$.

In Table 2, we compare the exact solution,approximate solution by our method and solution in[12]. Also values of $u(x)$ for $\alpha = 0.98$ and $\alpha = 0.98$. From Figure 2, we see that as α approaches 1, the numerical solution converges to that of integer-order differential equation.

5 Figures and Tables

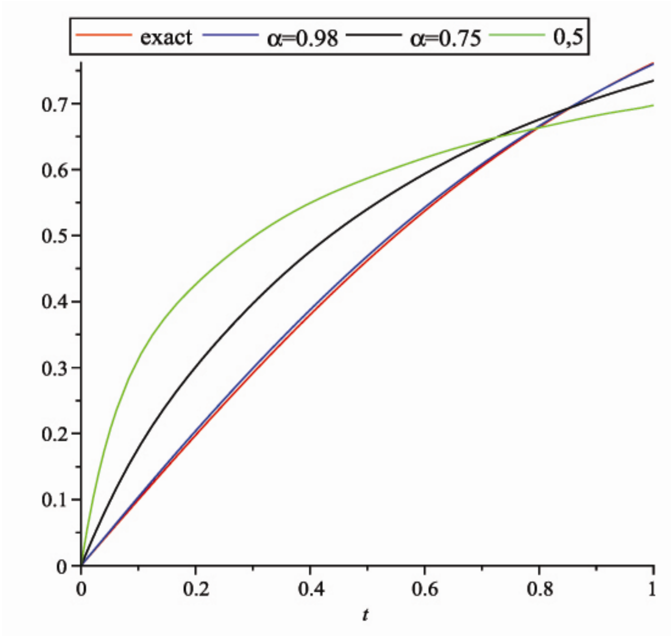

Figure. 1: Comparison of u(x) for m = 10 and with α = 0.5, 0.75, 0.98, 1, for Example 1

Table 1: Comparison between, the numerical solution using [12] and the approximate solution using our proposed method at $\alpha = 1$ for Example 1

X	exact	Present method	method in	$\alpha = 0.98$	$\alpha = 0.75$
0.1	0.099667	0.099667	0.099668	0.103687	0.177702
0.2	0.197375	0.197375	0.197375	0.203843	0.300755
0.3	0.291312	0.291312	0.291313	0.298710	0.396847
0.4	0.379948	0.379948	0.379944	0.387358	0.475789
0.5	0.462117	0.462117	0.462078	0.468723	0.539956
0.6	0.537049	0.537049	0.536857	0.542338	0.593448
0.7	0.604367	0.604367	0.603631	0.608056	0.638465
0.8	0.664036	0.664036	0.661706	0.665936	0.675767
0.9	0.716297	0.716297	0.709919	0.716423	0.707735
1	0.761594	0.761594	0.746032	0.760027	0.734731

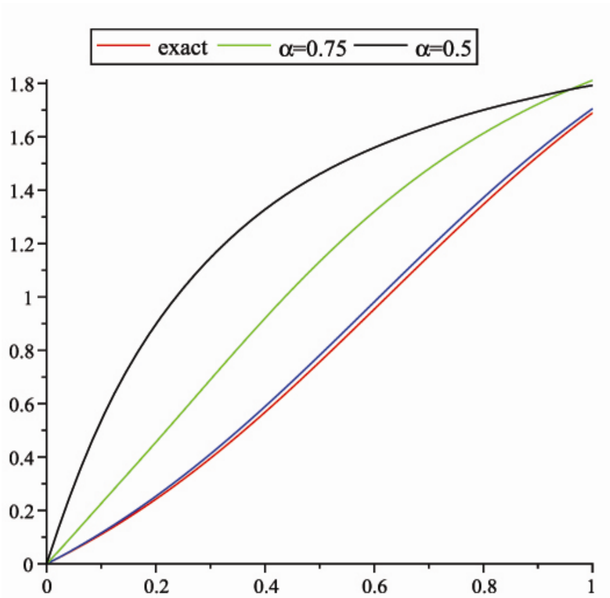

Figure. 2: Comparison of $u(x)$ for m = 10 and with α = 0.5, 0.75, 0.98, 1, for Example 2

Table 2: Comparison between, the numerical solution using [12] and the approximate solution using our proposed method at $\alpha = 1$ for Example 2.

X	exact	Present method	method in [12]	a = 0.98	a = 0.75
0.1	0.110295	0.110298	0.110294	0.1154867087	0.2259844123
0.2	0.241976	0.241980	0.241965	0.252793	0.455368
0.3	0.395104	0.395109	0.395106	0.411317	0.689809
0.4	0.567812	0.567817	0.568115	0.589043	0.919952
0.5	0.756014	0.756019	0.757564	0.781108	1.132044
0.6	0.953566	0.95357	0.958259	0.980806	1.319546
0.7	1.152948	1.152954	1.163459	1.180229	1.479780
0.8	1.346363	1.346368	1.365240	1.371442	1.61272
0.9	1.526911	1.526916	1.554960	1.547936	1.72237
1	1.689498	1.689502	1.723810	1.705192	1.811774

6 Conclusions

The properties of the Legendre polynomials are used to reduce the fractional diffusion equation to the solution of system of nonlinear equations. From the solutions obtained using the suggested method we can conclude that these solutions are in excellent agreement with the already existing ones. ([12], [14], [15], [7]).

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