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Lie Ring of $4-$move Invariant Group $R_4(L)$ and Kawauchi's $4-$move Conjecture

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Abstract

We study the invariants of 4−move defined in [5], and calculate Lie ring of the group $R_4(L)$ in response to the question proposed by Kawauchi [8], are link- homotopic links 4-move equivalent? We test the strength of the invariant $\mathfrak{R}_4(L) = \pi_1(S\mathbb{L})/N$ over the nth Burnside group of links and then apply it on link "$\mathfrak{L}$", motivated by Askitas knot and propose it as a potential counter example to Kawauchi's question.

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1 Introduction

This work is motivated by the example of Askitas knot proposed in [2] as a counter example to Nakanishi's 4−move conjecture [9]. Askitas, suspected that $(2, 1)-$ cable of the figure eight knot as the simplest counter example to the Nakanishi's 4−move conjecture. Since every link can not be reduced to a

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trivial link by 4−move, in particular the linking matrix modulo 2 is preserved by 4−move. Kawauchi [7], later on, expressed his questions for links as follow:
(i) Is it true that if two links are link homotopic, then they are 4−move equivalent?
(ii) In particular, is it true that every 2−component link is 4−move equivalent to the trivial link of two components or to the Hopf link? In this paper we response to Kawauchi's above posed questions, the work was motivated by our initial effort to address Nakanishi's 4−move conjecture. Our effort of 4−move reduction of Askitas knot [2] was unsuccessful, and the knot still stands as a counter example of Nakanishi's conjecture. However, as a result, we created a link $\mathfrak{L}$, the double figure eight with an additional component (shown in Figure 1), which can not be reduced by 4-moves invariants. Our results of 4- moves reduction of the link $\mathfrak{L}$ conclude that the link $\mathfrak{L}$ is a potential counter example to Kawauchi's conjecture. We organize this paper in three main sections, the first section introduces essential results requisite for clear understanding of the topic. Section two presents the construction of link $\mathfrak{L}$ and the 4−move invariant $R_4(L)$ defined in [5], and the main results obtained for the link $\mathfrak{L}$. We conclude our results with some speculations for future research in the last section.

2 The Invariants

The rational moves and their invariants have been introduced and studied for their importance in reduction of knots and links. Such invariants for instance, the nth Burnside group of links was used by J. Przytycki and M. Dabkowski [4] to solve some long standing problems in the classical knot theory.

Definition 2.1. *The nth Burnside group of a link is the quotient of the fundamental group of the double branched cover of S with the link as the branch set divided by all relations of the form $\omega^n = 1$. Succinctly:*

$$BL(n) = \pi_1(M_L^{(2)})/\,(\omega^n)$$

Notice that for the trivial link of k components, T_k , one has $B_{T_k}(n) = B(k-1,n)$, where $B(k-1,n)$ is the classical Burnside group of $k-1$ generators and exponent n. However; this method cannot be used in the case when the abelianization of the nth Burnside group, i.e. $H_1(M_L^{(2)}, Z_n)$ is a cyclic group. That is, for the 4th Burnside group of a link L of 2 components, it was shown that $B_L(4)$ is a quotient of $\mathbb{Z}_4$. Therefore, $B_L(4)$ caries no more information about $4-$ move equivalence classes as the group of 4-coloring. Therefore, the Nakanishi $4-$ move conjecture remains open as well as 2-component version of the Kawauchi 4-move question. Later on, Dabkowski and Sahi [5] defined a new invariant of links $R_4(L)$ that is preserved by $4-$ moves, and that can be used to answer several problems concerning $4-$moves and is potentially stronger than the 4th Burnside group of a link L of 2 components. Here are some main results concerning $R_4(L)$, invariant of $4-$moves:

Definition 2.2. *Let L be a link and $\pi(S^3 \setminus L)$ denote fundamental group of the complement of L in S. We define: $R_4(L) = \pi_1(S^3 \setminus L)/N$, where*

$$N =<< (a\omega b\omega^{-1})^2(\omega b\omega^{-1}a)^{-2} | \omega \in \pi_1(S^3 \setminus L), a, b \in X^{\pm 1} >>$$

is the normal subgroup of $\pi(S^3 \setminus L)$ generated by

$$R = (a\omega b\omega^{-1})^2(\omega b\omega^{-1}a)^{-2} | \omega \in \pi_1(S^3 \setminus L), a, b \in X^{\pm 1}.$$

Theorem 2.3. [6] *$R_4(L)$ is an invariant of link L which is preserved by $4-$moves.*

Theorem 2.4. *Let T_2 be a trivial link of two components and $a, b \in x_1^{\pm 1}, x_2^{\pm 1}$, then $\mathfrak{R}_4(T_2) =< x^1, x^2 | a^\epsilon bab = baba^\epsilon, ab^4 = b^4a, a^2b^2 = b^2a^2, \epsilon = \pm 1 >$. The main tool which they used to derive further results about the invariant $R_4(L)$ was a technique of the associated Lie Rings of a group introduced by W. Magnus* [11].

Theorem 2.5. [6] *The associated Lie ring of $R_4(T_2)$ decomposes into the homogenous terms [2] as follows:*

$$L(\mathfrak{R}_4(T_2)) = L_1(T_2) \oplus L_2(T_2) \oplus L_3(T_2) = \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2$$

[2]Dabkowski and Sahi proved much more general result for trivial links of m components, however we only restrict here to a Corollary of their results.

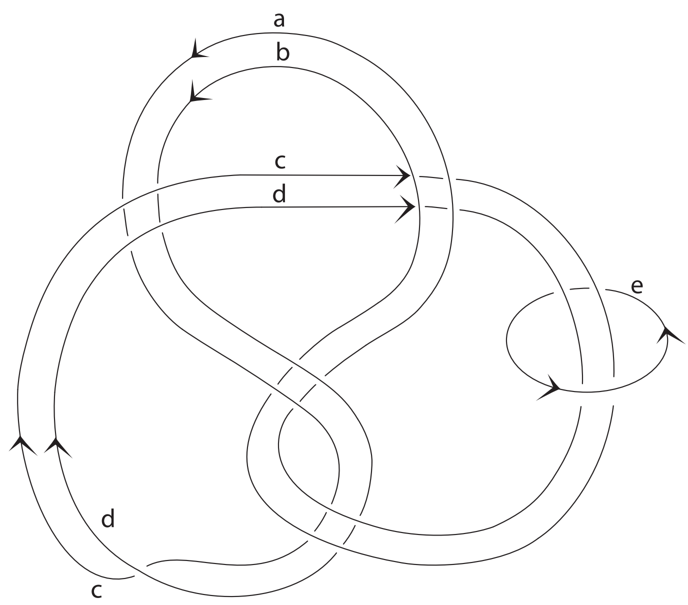

Figure 1: Double Figure Eight with an additional component

Here we note that for the 4th Burnside group of a 2−component link L, the group $B_L(4)$ is a quotient of $\mathbb{Z}_4$ as we mentioned earlier. Since $L_3(T_2) \neq 0$, the invariant $R_4(L)$ is potentially stronger for links of 2− components than $B_L(4)$.

3 The Link: Double figure Eight with an additional component

Problem:

Is the link $\mathfrak{L}$ (Double figure Eight with an additional component) shown in "figure1", 4− move reducible to the trivial link or Hopf link?

As mentioned before, the example of the above shown link $\mathfrak{L}$ was motivated by the example given in [2]. The link $\mathfrak{L}$ is obtained by adding an additional component to double figure eight knot and has 21 crossings. Our first approach towards the link $\mathfrak{L}$ was crossing reduction. We apply 4-moves and isotopy to reduce 21 crossing to 11 crossings or less, as we have well established results for all links up to 11 crossing are 4-moves reducible to trivial link or Hopf link

[6]. Practically, it was not easy to handle the link when applying isotopy and $4-$moves, since the number of crossings being so large, 21. Therefore it was necessary to move forward with option of using invariant $R_4(L)$ for the link $\mathfrak{L}$. In this regard, we first use the commutator calculus to calculate the associated Lie ring of the group $R_4(L)$ for the link $\mathfrak{L}$. We organize our results as a sequence of propositions and lemmas and present the meticulous calculations in the following sections.

Theorem 3.1. (Wirtinger) *Let D be an oriented diagram of a link L in S^3, then the group $\prod_1(D)$ is an invariant of a link L and $\prod_1(D) \cong (S^3 \setminus L)$, where $\prod_1(D) =< x_1, x_2, \ldots, x_n \,|\, r_1, r_2, \ldots, r_m >$ is the group associated to the diagram D of link L and $\prod_1(S^3 \setminus L)$ denotes the fundamental group of the complement of link L in S^3.*

It was also shown by Wirtinger, one of the relations can be always dropped from the presentation of $\prod_1(D)$, as it is a consequence of the other relations and the group defined by the above presentation does not depend (up to isomorphism) on the orientation of the diagram D.

Theorem 3.2. *Let L be the link shown in "Figure 1", then the fundamental group of the complement of L in S^3 admits the following presentation:*

$$\pi_1(S^3 \setminus L) =< a, b, c, d, e \,|\, R_1, R_2, R_3, R_4, R_4 >,$$

where

$$\begin{aligned}
R_1 &= [[(ab)^{-1}, (dc)^{-1}]dc, e] \\
R_2 &= [[dc, (eab)^{-1}, b^{-1}]bd^{-1}] \\
R_3 &= [[dc, (eab)^{-1}]d, a^{-1}]ac^{-1} \\
R_4 &= [e^{-1}[e^{-1}, ab][ab, (dc)^{-1}], c^{-1}]cb^{-1} \\
R_5 &= [e^{-1}[e^{-1}, ab][ab, (dc)^{-1}], d^{-1}]da^{-1}
\end{aligned}$$

We label the arcs of the link L by the generators of the free group $F_5 =< a, b, c, d, e \,|\, - >$ as it is shown in Figure 1. By using Wirtinger Theorem, we derive the presentation as described above in the statement. The relations are calculated by the chain of lemmas:

Lemma 3.3. *Let G be a group and $x, y, z \in G$. Then the following identities hold:*

$$\begin{array}{lcl} [xy, z] & = & [x, z][x, z, y][y, z], \\ xy & = & yx[x, y], \\ [x^{-1}, y] & = & [x, y]^{-1}[y, x, x^{-1}], \\ [x, y^{-1}] & = & [x, y]^{-1}[y, x, y^{-1}], \\ [x^{-1}, y^{-1}] & = & [x, y][x, y, (yx)^{-1}] \end{array}$$

Proof. The first two identities are the standard commutator identities which one can, for instance, find in [12]. The last three identities follow by simple computations.

$[x^{-1}, y] = xy^{-1}x^{-1}y = x(y^{-1}x^{-1}yx)x^{-1} = x[y, x]x^{-1} = xx^{-1}[x, y]^{-1}[y, x, x^{-1}]$

$[x, y^{-1}] = x^{-1}yxy^{-1} = y(y^{-1}x^{-1}yx)y^{-1} = y[y, x]y^{-1} = yy^{-1}[y, x][y, x, y^{-1}]$
$= [x, y]^{-1}[y, x, y^{-1}]$

$[x^{-1}, y^{-1}] = xyx^{-1}y^{-1} = yx(x^{-1}y^{-1}xy)(yx)^{-1} = yx[x, y](yx)^{-1} = [x, y][x, y, (yx)^{-1}]$

This finishes our argument. □

Corollary 3.4. *Let G be a group with $x, y, z \in G$ and let $\gamma_1 \geq \gamma_2 \geq \gamma_3 \ldots$ be the lower central series of G. Then*

$$\begin{array}{lcl} [xy, z] & \equiv_{\gamma_4} & [x, z] + [x, z, y] + [y, z] \\ [x^{-1}, z] & \equiv_{\gamma_4} & [x, y]^{-1} - [y, x, x] \\ [x, y^{-1}] & \equiv_{\gamma_4} & [x, y]^{-1} - [y, x, y] \\ [x^{-1}, y^{-1}] & \equiv_{\gamma_4} & [x, y] - [x, y, y] - [x, y, x] \end{array}$$

A proof follows directly from Lemma 3.3 since, for instance for the second identity, we have:

$$\begin{array}{lclcl} [x, y^{-1}] & = & [x, y]^{-1}[x, y, x^{-1}] & \equiv_{\gamma_4} & [x, y]^{-1} + [[y, x], x]^{-1} \\ & = & [x, y]^{-1} + [[y, x], x^{-1}] + [x, [y, x], x^{-1}] & \equiv_{\gamma_4} & [x, y]^{-1} - [y, x, x]. \end{array}$$

Analogous calculations prove the remaining identities.

Lemma 3.5. *Let $R_1 = [[(ab)^{-1}, (dc)^{-1}]dc, e] \in F_5$. Then we have:*

$$\begin{array}{lcl} R_1 & \equiv_{\gamma_4} & -[d, a, e] - [c, a, e] - [d, b, e] - [c, b, e] + [d, c, e] - [e, c, d] + [e, d]^{-1} + [e, c]^{-1} \\ & = & -e_{19} - e_{31} - e_{23} - e_{35} + e_{26} - e_{11} + [e, d]^{-1}[e, c]^{-1} \end{array}$$

We apply the commutator identities given in Lemma 3.3 and the result of Lemma 3.4 We have:

$$\begin{aligned}
R_1 &= [[(ab)^{-1},(dc)^{-1}]dc,e] = [(ab)^{-1},(dc)^{-1},e][[(ab)^{-1},(dc)^{-1}],e,dc][dc,e] \\
&\equiv_{\gamma_4} [(ab)^{-1},(dc)^{-1},e] + [dc,e] \equiv_{\gamma_4} [ab,dc,e] + [d,e] + [d,e,c] + [c,e] \\
&\equiv_{\gamma_4} [a,d,e] + [a,c,e] + [b,d,e] + [b,c,e] + [d,e] + [d,e,c] + [c,e]
\end{aligned}$$

Now, we use the Jacobi identity:

$$[d,e,c] = -[e,c,d] - [c,d,e] = [d,c,e] - [e,c,d]$$

and we obtain;

$$R_1 \equiv_{\gamma_4} -[d,a,e]-[c,a,e]-[d,b,e]-[c,b,e]+[d,c,e]-[e,c,d]+[e,d]^{-1}+[e,c]^{-1},$$

which finishes our proof. Analogously, we have:

Lemma 3.6. *Let* $R_2 = [[dc,(eab)^{-1}],b^{-1}]bd^{-1} \in F_5$. *Then*

$$R_2 \equiv_{\gamma_4} ([d,b,e]-[e,b,d]+[d,a,b]+[d,b,b]+[c,b,e]-[e,b,c]+[c,a,b]+[c,b,b])+bd^{-1}.$$

We also have:

Lemma 3.7. *Let* $R_3 = [[dc,(eab)^{-1}]d,a^{-1}]ac^{-1} \in F_5$. *Then*

$$\begin{aligned}
R_3 \equiv_{\gamma_4} & [d,a,e] - [e,a,d] + 2[d,a,a] + [d,a,b] - [b,a,d] + [c,a,e] \\
& - [e,a,c] + [c,a,a] + [c,a,b] - [b,a,c] + [d,a]^{-1}ac^{-1}.
\end{aligned}$$

Moreover, we have:

Lemma 3.8. *Let* $R_4 = [[e^{-1},ab][ab,(dc)^{-1}],c^{-1}]cb^{-1} \in F_5$. *Then*

$$R_4 \equiv_{\gamma_4} ([e,a,c]+[e,b,c]-[e,c,c]-[e,c,e]-[c,a,c]-[d,a,c]-[d,b,c]-[c,b,c])+[e,c]cb^{-1}.$$

Finally, we obtain:

Lemma 3.9. *Let* $R_5 = [e^{-1}[e^{-1}, ab][ab, (dc)^{-1}], d^{-1}]da^{-1} \in F_5$. *Then*

$R_5 \equiv_{\gamma_4} ([e, a, d]+[e, b, d]-[e, d, d]-[e, d, e]-[d, a, d]-[c, a, d]-[d, b, d]-[c, b, d])+[e, d]da^{-1}$.

That completes the proof.

Corollary 3.10. *Let* R_1, R_2, R_3, R_4, R_5 *be the relations defined in Theorem 3.2. Then*

$$R_1, \equiv_{\gamma_2} 0;\ R_2 \equiv_{\gamma_2} b-d;\ R_3 \equiv_{\gamma_2} a-c;\ R_4 \equiv_{\gamma_2} c-b;\ R_5 \equiv_{\gamma_2} d-a$$

the relations R_2, R_3 *and* R_4 *are linearly independent over* $\mathbb{Z}$ *in* γ_1/γ_2, *where;*

$$R_5 \equiv_{\gamma_2} R_2^{-1}R_4^{-1}R_3^{-1} = (R_3R_4R_2)^{-1} and\, \gamma_1/\gamma_2 \equiv \mathbb{Z}\oplus\mathbb{Z}.$$

Proof. As it is known [12], γ_1/γ_2 is generated by a, b, c, d, e. We order them in the alphabetic order $a < b < c < d < e$. Therefore, we obtain the following matrix of the presentation for γ_1/γ_2 : Now, let

	a	b	c	d	e
R_1	0	0	0	0	0
R_2	0	1	0	−1	0
R_3	0	1	−1	0	0
R_4	0	−1	1	0	0
R_5	−1	0	0	1	0

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We observe that, $rank(\mathbf{M}) = 3$. Thus, we obtain,

$$\gamma_1/\gamma_2 \equiv \mathbb{Z}\oplus\mathbb{Z}.$$

Moreover, we have

$$R_5 \equiv_{\gamma_2} (R_3R_4R_2)^{-1} \equiv_{\gamma_2} -(c-b)-(a-c)-(b-d) = d-a.$$

$$
\begin{array}{lll}
[R_i, x] & \equiv_{\gamma_3} & 0,\ i = 2, 3, 4; x \in a, b, c, d, e \\
R_1 & \equiv_{\gamma_3} & 0 \\
R_5R_4R_3R_2 & \equiv_{\gamma_3} & 0
\end{array}
$$

This finishes our proof. □

Using the algorithm for finding presentation of the successive terms of the quotients of the lover central series given in [9] we conclude that the matrix of the presentation of γ_2/γ_3 has the following rows:

Let $e_1 = [e, a]$, $e_2 = [e, b]$, $e_3 = [e, c]$, $e_4 = [e, d]$, $e_5 = [d, a]$, $e_6 = [d, b]$, $e_7 = [d, c]$, $e_8 = [c, a]$, $e_9 = [c, b]$, $e_{10} = [b, a]$ with $e_1 < e_2 < \ldots < e_{10}$ be a basis of γ_2/γ_3. The following result holds:

Lemma 3.11. *Let R_1, R_2, R_3, R_4, R_5 be the relations defined in Theorem 3.2. Then:*

$$\begin{array}{lll lll}
R_1 & \equiv_{\gamma_3} & -e_3 - e_4 & [R_2, a] & \equiv_{\gamma_3} & -e_5 + e_{10} \\
[R_2, b] & \equiv_{\gamma_3} & -e_6 & [R_2, c] & \equiv_{\gamma_3} & -e_7 - e_9 \\
[R_2, d] & \equiv_{\gamma_3} & -e_6 & [R_2, e] & \equiv_{\gamma_3} & -e_2 + e_4 \\
[R_3, a] & \equiv_{\gamma_3} & -e_8 & [R_3, b] & \equiv_{\gamma_3} & -e_9 - e_{10} \\
[R_3, c] & \equiv_{\gamma_3} & -e_8 & [R_3, d] & \equiv_{\gamma_3} & -e_5 + e_7 \\
[R_3, e] & \equiv_{\gamma_3} & -e_1 + e_3 & [R_4, a] & \equiv_{\gamma_3} & e_8 - e_{10} \\
[R_4, b] & \equiv_{\gamma_3} & e_9 & [R_4, c] & \equiv_{\gamma_3} & e_9 \\
[R_4, d] & \equiv_{\gamma_3} & e_6 - e_7 & [R_4, e] & \equiv_{\gamma_3} & e_2 - e_3 \\
R_5R_3R_4R_2 & \equiv_{\gamma_3} & e_3 + e_4 - e_5 & & &
\end{array}$$

Proof. The proof follows simple calculations using commutator calculus. □

Corollary 3.12. *Let R_1, R_2, R_3, R_4, R_5 be the relations defined in Theorem 3.2. Then:*

$$(\gamma_2/\gamma_3) \otimes \mathbb{Z}_2 \equiv \mathbb{Z}_2.$$

Proof. Applying Lemma 3.11 we obtain the "Relation Matrix" of the presentation for (γ_2/γ_3), see Table 1 on next page.

From these relations, we find that:

$$\begin{array}{lll}
\gamma_2/\gamma_3 & = & \{e_i,\ 1 \le i \le 10 \mid -e_1 - e_4, e_2 - e_4, -e_3 - e_4, 2e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\} \\
& = & \{e_1, e_2, e_3, e_4 \mid -e_1 - e_4, e_2 - e_4, -e_3 - e_4, 2e_4\} \\
& = & \{e_1 \mid 2e_4\} \\
& \equiv & \mathbb{Z}_2
\end{array}$$

Therefore, we have: $(\gamma_2/\gamma_3) \otimes \mathbb{Z}_2 \equiv \mathbb{Z}_2$. This finishes our proof here. □

Table 1: Relation Matrix

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
R_1	0	0	-1	-1	0	0	0	0	0	0
$[R_2, a]$	0	0	0	0	-1	0	0	0	0	1
$[R_2, b]$	0	0	0	0	0	-1	0	0	0	0
$[R_2, c]$	0	0	0	0	0	0	-1	0	-1	0
$[R_2, d]$	0	0	0	0	0	-1	0	0	0	0
$[R_2, e]$	0	-1	0	1	0	0	0	0	0	0
$[R_3, a]$	0	-1	0	1	0	0	0	0	0	0
$[R_3, b]$	0	0	0	0	0	0	0	0	-1	-1
$[R_3, c]$	0	0	0	0	0	0	0	-1	0	0
$[R_3, d]$	0	0	0	0	-1	0	0	0	0	0
$[R_3, e]$	-1	0	1	0	0	0	0	0	0	0
$[R_4, a]$	0	0	0	0	0	0	0	0	0	-1
$[R_4, b]$	0	0	0	0	-1	0	0	0	1	0
$[R_4, c]$	0	0	0	0	-1	0	0	0	1	0
$[R_4, d]$	0	0	0	0	-1	1	-1	0	0	0
$[R_4, e]$	0	1	-1	0	0	0	0	0	0	0
$R_5R_3R_4R_2$	0	0	1	1	-1	0	0	0	0	0

Since the rank of the matrix of the presentation of γ_2/γ_3 is 10 over $\mathbb{R}$, we need to find 10 linearly independent rows of the matrix of relations and use them to express the remaining 8 relations. We make the following choices for our relations:

$$\begin{array}{lcl|lcl|lcl}
R_1 & = & -e_3 - e_4 & [R_2, c] & = & -e_7 - e_9 & [R_3, d] & = & -e_5 + e_7 \\
[R_2, a] & = & -e_5 - e_{10} & [R_2, e] & = & -e_2 + e_4 & [R_3, e] & = & -e_1 + e_3 \\
[R_2, b] & = & -e_6 & [R_3, a] & = & -e_8 & [R_4, b] & = & e_9 \\
[R_4, e] & = & e_2 - e_3 & & & & & &
\end{array}$$

Lemma 3.13. *Let R_1, R_2, R_3, R_4, R_5 be the relations defined in Theorem 3.2 . Then the relations given above are linearly independent over $\mathbb{Z}_2$ and* *moreover we have:*

$$\begin{array}{ll}
[R_2,d][R_2,b]^{-1} & \equiv_{\gamma_3} 0\\
[R_3,c][R_3,a]^{-1} & \equiv_{\gamma_3} 0\\
[R_4,c][R_4,b]^{-1} & \equiv_{\gamma_3} 0\\
[R_3,b][R_2,c]^{-1}[R_3,d]^{-1}[R_2,a] & \equiv_{\gamma_3} 0\\
[R_4,d][R_4,b]^{-1}[R_2,c]^{-1}[R_2,b] & \equiv_{\gamma_3} 0\\
[R_4,a][R_4,b]^{-1}[R_2,c]^{-1}[R_3,d]^{-1}[R_2,a][R_3,a] & \equiv_{\gamma_3} 0\\
(R_5R_3R_4R_2)[R_4,b]^{-1}[R_2,c]^{-1}[R_3,d]^{-1}R_1 & \equiv_{\gamma_3} 0
\end{array}$$

Proof. The proof follows by direct calculations and the result of Lemma 3.11.

Now, we start calculating the relations $mod\gamma_4$. We start by fixing the basis for γ_3/γ_4 of third commutators:

$$\begin{array}{llllllllllll}
e_1 & = & [e,a,a] & e_{11} & = & [e,c,d] & e_{21} & = & [d,b,c] & e_{31} & = & [c,a,e]\\
e_2 & = & [e,a,b] & e_{12} & = & [e,c,e] & e_{22} & = & [d,b,d] & e_{32} & = & [c,b,b]\\
e_3 & = & [e,a,c] & e_{13} & = & [e,d,d] & e_{23} & = & [d,b,e] & e_{33} & = & [c,b,c]\\
e_4 & = & [e,a,d] & e_{14} & = & [e,d,e] & e_{24} & = & [d,c,c] & e_{34} & = & [c,b,d]\\
e_5 & = & [e,a,e] & e_{15} & = & [d,a,a] & e_{25} & = & [d,c,d] & e_{35} & = & [c,b,e]\\
e_6 & = & [e,b,b] & e_{16} & = & [d,a,b] & e_{26} & = & [d,c,e] & e_{36} & = & [b,a,a]\\
e_7 & = & [e,b,c] & e_{17} & = & [d,a,c] & e_{27} & = & [c,a,a] & e_{37} & = & [b,a,b]\\
e_8 & = & [e,b,d] & e_{18} & = & [d,a,d] & e_{28} & = & [c,a,b] & e_{38} & = & [b,a,c]\\
e_9 & = & [e,b,e] & e_{19} & = & [d,a,e] & e_{29} & = & [c,a,c] & e_{39} & = & [b,a,d]\\
e_{10} & = & [e,c,c] & e_{20} & = & [d,b,b] & e_{30} & = & [c,a,d] & e_{40} & = & [b,a,e]
\end{array}$$

In order to derive a presentation of γ_3/γ_4 we use the algorithm given in [9]. That is we first find all the relations: $[R_1,x]mod\gamma_4$, $[[R_2,a],x]mod\gamma_4$, $[[R2,b],x]mod\gamma_4$, $\ldots$, $[[R4,b],x]mod\gamma_4$, where $x \in a,b,c,d,e$. This way we obtain the first 45 relations for the matrix of the presentation of γ_3/γ_4. Now, we find $[[R2,a],x]mod\gamma_4$. Applying commutator calculus, we obtain the following matrix of the presentation for γ_3/γ_4:

Hence, it follows that:

$$L_3(\mathfrak{L}) = \mathbb{Z}_2^2 = L_3(T_2)$$

Therefore, we conclude our result and complete this section here. $\square$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
$[R_3, e]R_1$	-1	0	0	-1	0	0	0	0	0	0
$[R_2, e]$	0	-1	0	1	0	0	0	0	0	0
R_1	0	0	-1	-1	0	0	0	0	0	0
$[R_4, e](R_1)^{-1}[R2, e]$	0	0	0	2	0	0	0	0	0	0
$[R_2, a][R_3, b][R_4, b]$	0	0	0	0	-1	0	0	0	0	0
$[R_2, d]$	0	0	0	0	0	-1	0	0	0	0
$[R_2, c][R_4, b]$	0	0	0	0	0	-1	0	0	0	0
$[R_3, c]$	0	0	0	0	0	0	0	-1	0	0
$[R_4, b]$	0	0	0	0	0	0	0	0	1	0
$[R_3, b][R_4, b]$	0	0	0	0	0	0	0	0	0	-1
$[R_2, b]([R_2, d])^{-1}$	0	0	0	0	0	0	0	0	0	0
$[R_3, a]([R_3, c])^{-1}$	0	0	0	0	0	0	0	0	0	0
$[R_3, d][R_2, c][R_4, b]$ $([R_2, a][R_3, b][R_4, b])^{-1}$	0	0	0	0	0	0	0	0	0	0
$[R_4, a]([R_3, b][R_4, b])^{-1}[R_3, c]$	0	0	0	0	0	0	0	0	0	0
$[R_4, c][R_4, b]^{-1}$	0	0	0	0	0	0	0	0	0	0
$[R_4, d]([R_2, c][R_4, b])^{-1}[R_2, d]$	0	0	0	0	0	0	0	0	0	0
$R_5R_3R_4R_2([R_2, a][R_3, b][R_4, b])^{-1}R_1$	0	0	0	0	0	0	0	0	0	0

4 Conclusion

The combinatorial approach to reduce the link L by 4−moves and isotopy to a link of 11 crossings or less is not convenient at all as number of crossings go beyond control. The 4-moves invariant $R_4(L)$ defined in [5], is potentially stronger than nth Burnside group of links; as the $B_L(4)$ deficits information about 4−move equivalence classes. As we have mentioned earlier, the example of the link $\mathfrak{L}$ was motivated by the example given in [2], that is still an open problem therefore, we conclude that the link $\mathfrak{L}$, serves as a counterexample for Kawauchi's question. The results we obtained as an effect of calculations of the associated Lie ring of the group $R_4(L)$ only show that using the commutator calculus we cannot deduce if $R_4(L) \not\cong R_4(T_2)$. This does however implies that other invariants like, $K_4^{(n)}(L)$ and $N_4^{(n)}(L)$ [6] can be used to show that if $\mathfrak{L}$ reduces to the trivial link with 2 components or not. The challenge in this context is to find the convenient presentations for these groups, once knowing their presentations computer algebra systems (for instance GAP) can be used to make the necessary computations. For now, we leave these ideas open and plan to investigate more techniques from the combinatorial group theory in the near future.

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