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The Join Mapping of two Stratified Graphs

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Abstract

Using the mappings u_1 and u_2 that uniquely define [9] two stratified graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, we define the mapping $u_1 \oplus u_2$. This mapping is used in further research to define the least upper bound of stratified graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. The upper bound helps us in future research to prove the closure under union set operation of stratified languages, a family of languages generated by stratified graphs. A few properties, including the associativity of the operation $\oplus$ are proved.

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1 Introduction

The concept of labeled stratified graph (shortly, a *SG* or stratified graph) was introduced in [7] as a method of knowledge representation and it is obtained by incorporating the concept of *labeled graph* into an algebraic environment given by a tuple of components, which are obtained applying several

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concepts of *universal algebra*. The existence of this structure is proved in [8] and various algebraic properties are presented in [8], [9], [10] and [11]. The inference based on *SGs* is described in [10]. An application of this structure in the communication semantics and graphical image generation are presented in [10]. A special kind of reasoning, hierarchical reasoning and its application to image synthesis are presented in [11]. In [5], [6] we show that we can generate formal languages by means of the stratified graphs, thus obtaining a new mechanism to generate formal languages.

Let M be an arbitrary nonempty set. We consider the set B given by

$$B = \bigcup_{n \geq 0} B_n$$

where

$$\begin{cases} B_0 = M \\ B_{n+1} = B_n \cup \{\sigma(x_1, x_2) \mid (x_1, x_2) \in B_n \times B_n\}, n \geq 0 \end{cases}$$

and $\sigma(x_1, x_2)$ is the word $\sigma x_1 x_2$ over the alphabet $\{\sigma\} \cup M$. The pair $\overline{M} = (B, \sigma)$ is a Peano σ-algebra over M ([1], [2], [3], [4]).

If f and g are two mappings then we write $f \preceq g$ if $dom(f) \subseteq dom(g)$ and $f(x) = g(x)$ for every $x \in dom(f)$.

A binary relation ρ over the set S is a subset $\rho \subseteq S \times S$. The set of all binary relations is the power set $2^{S \times S}$. There is a classical binary operation for binary relations. This is denoted by

$$\circ : 2^{S \times S} \times 2^{S \times S} \longrightarrow 2^{S \times S}$$

and is defined as follows:

$$\rho_1 \circ \rho_2 = \{(x, y) \in S \times S \mid \exists z \in S : (x, z) \in \rho_1, (z, y) \in \rho_2\}$$

We consider the mapping $prod_S : dom(prod_S) \longrightarrow 2^{S \times S}$, where

$$dom(prod_S) = \{(\rho_1, \rho_2) \in 2^{S \times S} \times 2^{S \times S} \mid \rho_1 \circ \rho_2 \neq \emptyset\}$$

and $prod_S(\rho_1, \rho_2) = \rho_1 \circ \rho_2$ for every $(\rho_1, \rho_2) \in dom(prod_S)$. The pair $(2^{S \times S}, prod_S)$ becomes a partial algebra. We denote by $u \in R(prod_S)$ the following property: $u : dom(u) \longrightarrow 2^{S \times S}$, $dom(u) \subseteq dom(prod_S)$, $u(\rho_1, \rho_2) = \rho_1 \circ \rho_2$ for every $(\rho_1, \rho_2) \in dom(u)$. If $u \in R(prod_S)$ then we denote by $Cl_u(T_0)$ the closure of T_0 in $2^{S\times S}$. This is the least subset of $2^{S\times S}$ that contains T_0 and is closed with respect to u.

The concept of labeled graph is a basic one for the concept of stratified graph. By a *labeled graph* we understand a tuple $G = (S, L_0, T_0, f_0)$, where S is a finite set of nodes, L_0 is a set of elements named *labels*, T_0 is a set of binary relations on S and $f_0 : L_0 \longrightarrow T_0$ is a surjective function.

We denote by $\mathcal{L}_{lg}$ the set of all labeled graphs. Consider $G_1 = (S_1, L_{01}, T_{01}, f_{01}) \in \mathcal{L}_{lg}$ and $G_2 = (S_2, L_{02}, T_{02}, f_{02}) \in \mathcal{L}_{lg}$. We write $G_1 \sqsubseteq G_2$ if $S_1 \subseteq S_2$, $L_{01} \subseteq L_{02}$ and $f_{01}(a) \subseteq f_{02}(a)$ for every $a \in L_{01}$. The relation $\sqsubseteq$ is a partial order as proved in [13].

Also in [13] we defined the mapping

$$f_{01} \sqcup f_{02} : L_{01} \cup L_{02} \longrightarrow T_{01} \cup T_{02} \cup \{\rho \mid \exists \mu \in T_{01}, \theta \in T_{02} : \rho = \mu \cup \theta\}$$

as follows:

$$(f_{01} \sqcup f_{02})(a) = \begin{cases} f_{01}(a) & \text{if} \quad a \in L_{01} \setminus L_{02} \\ f_{02}(a) & \text{if} \quad a \in L_{02} \setminus L_{01} \\ f_{01}(a) \cup f_{02}(a) & \text{if} \quad a \in L_{01} \cap L_{02} \end{cases}$$

Consider a nonempty set $L_0 \subseteq M$. We denote $L \in Initial(L_0)$ if the following two conditions are satisfied:

1. $L_0 \subseteq L \subseteq \overline{M}$;
2. if $\sigma(\alpha, \beta) \in L$, $\alpha \in \overline{M}$, $\beta \in \overline{M}$ then $\alpha \in L$ and $\beta \in L$.

A *labeled stratified graph* $\mathcal{G}$ over G (shortly, *stratified graph* or *SG*) is a tuple (G, L, T, u, f) where

- $G = (S, L_0, T_0, f_0)$ is a labeled graph
- $L \in Initial(L_0)$
- $u \in R(prod_S)$ and $T = Cl_u(T_0)$
- $f : (L, \sigma_L) \longrightarrow (2^{S\times S}, u)$ is a morphism of partial algebras such that $f_0 \preceq f$, $f(L) = T$ and if $(f(x), f(y)) \in dom(u)$ then $(x, y) \in dom(\sigma_L)$

2 The join mapping of two stratified graphs and its properties

We consider the labeled graphs

$$G_1 = (S_1, L_{01}, T_{01}, f_{01}) \in \mathcal{L}_{lg},\ \ G_2 = (S_2, L_{02}, T_{02}, f_{02}) \in \mathcal{L}_{lg}$$

and the labeled stratified graphs

$$\mathcal{G}_1 = (G_1, L_1, T_1, u_1, f^1) \in \mathcal{L}_{sg},\ \ \mathcal{G}_2 = (G_2, L_2, T_2, u_2, f^2) \in \mathcal{L}_{sg}$$

over G_1 and G_2 respectively.

In what follows we suppose that $S_1 \cap S_2 = \emptyset$. Without loss of generality we can suppose that

$$dom(u_1) \subseteq T_1 \times T_1,\ u_1 : dom(u_1) \longrightarrow T_1;$$

$$dom(u_2) \subseteq T_2 \times T_2,\ u_2 : dom(u_2) \longrightarrow T_2;$$

$T_1 \subseteq 2^{S_1 \times S_1}$, $T_2 \subseteq 2^{S_2 \times S_2}$ and $2^{S_1 \times S_1} \cap 2^{S_2 \times S_2} = \emptyset$, it follows that $T_1 \cap T_2 = \emptyset$.

Definition 2.1. *Take $S = S_1 \cup S_2$. We extend the mapping u_1 and u_2 as follows:*

$$\overline{u_1} : 2^{S \times S} \times 2^{S \times S} \longrightarrow 2^{S \times S}$$

$$\overline{u_1}(\rho_i, \rho_k) = \begin{cases} u_1(\rho_i, \rho_k) & if \quad (\rho_i, \rho_k) \in dom(u_1) \\ \emptyset & otherwise \end{cases}$$

$$\overline{u_2} : 2^{S \times S} \times 2^{S \times S} \longrightarrow 2^{S \times S}$$

$$\overline{u_2}(\omega_j, \omega_m) = \begin{cases} u_2(\omega_j, \omega_m) & if \quad (\omega_j, \omega_m) \in dom(u_2) \\ \emptyset & otherwise \end{cases}$$

We define

$$T_1 \Cup T_2 = \{\rho \cup \omega \mid \rho \in T_1 \cup \{\emptyset\}, \omega \in T_2 \cup \{\emptyset\}\} \setminus \{\emptyset\}$$

Remark 2.2. *Obviously we have $T_1 \Cup T_2 = T_2 \Cup T_1$.*

Proposition 2.3. *We consider the set* $N_0 = (f_{01} \sqcup f_{02})(L_{01} \cup L_{02}) \subseteq T_1^0 \Cup T_2^0$. *The sequence* $\{N_k\}_{k \geq 1}$ *defined recursively as follows:*

$$\begin{cases} N_1 = N_0 \cup \{\mu \in T_1^1 \Cup T_2^1 \mid \exists \rho_1 \cup \omega_1 \in N_0, \exists \rho_2 \cup \omega_2 \in N_0 : \\ \qquad \mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset\} \\ N_{k+1} = N_k \cup \{\mu \in T_1^{k+1} \Cup T_2^{k+1} \mid \exists \rho_1 \cup \omega_1 \in N_k, \exists \rho_2 \cup \omega_2 \in N_k : \\ \qquad \mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset\} \end{cases} \tag{1}$$

satisfies the following properties:

i1) $N_k \subseteq T_1^k \Cup T_2^k$, *for every* $k \geq 0$;

i2) $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_k \subseteq N_{k+1} \subseteq \ldots$

i3) There is $k_0 \geq 0$ *such that* $N_0 \subset \ldots \subset N_{k_0} = N_{k_0+1} = N_{k_0+2} = \ldots$

Proof. We have $N_0 = (f_{01} \sqcup f_{02})(L_{01} \cup L_{02}) = \{\rho \mid \exists a \in L_{01} \setminus L_{02} : \rho = f_{01}(a)\} \cup \{\rho \mid \exists a \in L_{02} \setminus L_{01} : \rho = f_{02}(a)\} \cup \{\rho \mid \exists a \in L_{01} \cap L_{02} : \rho = f_{01}(a) \cup f_{02}(a)\} \subseteq T_1^0 \cup T_2^0 \cup (T_1^0 \Cup T_2^0) \subseteq T_1^0 \Cup T_2^0$. Thus $i1$) is true for $k = 0$. Suppose that $i1$) is true for $k = m$ and we prove the property for $k = m + 1$. From (1) we obtain

$$N_{m+1} = N_m \cup \{\mu \in T_1^{m+1} \Cup T_2^{m+1} \mid \exists \rho_1 \cup \omega_1 \in N_m, \exists \rho_2 \cup \omega_2 \in N_m :$$

$$\mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset\}$$

therefore $N_{m+1} \subseteq (T_1^m \Cup T_2^m) \cup (T_1^{m+1} \Cup T_2^{m+1}) = T_1^{m+1} \Cup T_2^{m+1}$. Thus $i1$) is true for $k = m + 1$.

For every $k \geq 0$ we have $N_k \subseteq T_1^k \Cup T_2^k \subseteq T_1 \Cup T_2$ and the last set is a finite one because $S = S_1 \cup S_2$ is finite. Thus there is $k \geq 0$ such that $N_0 \subset \ldots \subset N_k = N_{k+1}$. Now, by induction on $p \geq 1$ we can verify that $N_k = N_{k+p}$. For $p = 1$ this property is true because $N_k = N_{k+1}$. Suppose that $N_k = N_{k+p}$ for $p = m$. We have

$$N_{k+m+1} = N_{k+m} \cup \{\mu \mid \exists \rho_1 \cup \omega_1 \in N_{k+m}, \exists \rho_2 \cup \omega_2 \in N_{k+m} :$$

$$\mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset\} = N_k \cup$$

$\{\mu \mid \exists \rho_1 \cup \omega_1 \in N_k, \exists \rho_2 \cup \omega_2 \in N_k : \mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset\} = N_{k+1} = N_k$

□

Proposition 2.4. *The sequence* $\{M_p\}_{p\geq 1}$ *defined as follows:*

$$\begin{cases} M_1 = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_0 \times N_0 \mid \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_1\} \\ M_{p+1} = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_{p-1} \times (N_p \setminus N_{p-1}) \cup (N_p \setminus N_{p-1}) \\ \qquad\qquad \times N_p : \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_{p+1}\} \end{cases} \tag{2}$$

satisfies the following property: either $M_1 = \emptyset$ *or there is* $k \geq 1$ *such that* $M_j \neq \emptyset$ *for every* $j \in \{1, \ldots, k\}$ *and* $M_j = \emptyset$ *for* $j \geq k+1$.

Proof. Suppose that $N_1 = N_0$. From the definition of N_1 we deduce that

$$\{\mu \mid \exists \rho_1 \cup \omega_1 \in N_0, \exists \rho_2 \cup \omega_2 \in N_0 : \mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset\} = \emptyset$$

therefore $M_1 = \emptyset$. If $N_1 \neq N_0$ then by Proposition 2.3 we deduce that there is $k \geq 1$ such that $N_0 \subset \ldots \subset N_k = N_{k+1} = N_{k+2} = \ldots$. In this case $M_j \neq \emptyset$ for $j \in \{1, \ldots, k\}$ and $M_j = \emptyset$ for $j \geq k+1$. □

Remark 2.5. *The rule by means of which the sequence* $\{M_p\}_{p\geq 1}$ *is obtained can be represented intuitively as in Figure 1.*

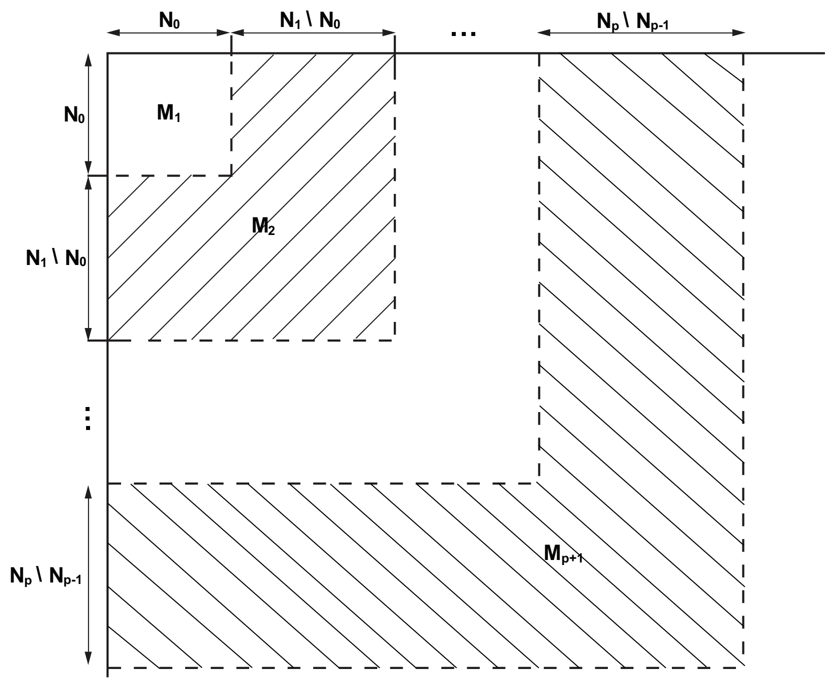

Figure 1: The sequence $\{M_p\}_{p\geq 1}$

Remark 2.6. *The relation (2) can be written also as in (3).*

$$\begin{cases} M_1 = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_0 \times N_0 \mid \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_1\} \\ M_{p+1} = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_{p-1} \times (N_p \setminus N_{p-1}) \cup (N_p \setminus N_{p-1}) \times \\ \times N_{p-1} \cup (N_p \setminus N_{p-1}) \times (N_p \setminus N_{p-1}) : \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_{p+1}\} \end{cases} \tag{3}$$

Remark 2.7. *For every $p \geq 1$ we have $M_p \subseteq (T_1 \Cup T_2) \times (T_1 \Cup T_2)$.*

Definition 2.8. *Consider the mappings $u_1 : dom(u_1) \longrightarrow T_1$ and $u_2 : dom(u_2) \longrightarrow T_2$, where $dom(u_1) \subseteq T_1 \times T_1$ and $dom(u_2) \subseteq T_2 \times T_2$. Consider the sequences $\{N_k\}_{k \geq 0}$ and $\{M_p\}_{p \geq 1}$ as in Propositions 2.3 and 2.4 respectively. Consider the number $k \geq 1$ such that $M_j \neq \emptyset$ for $j \in \{1, \ldots, k\}$ and $M_j = \emptyset$ for $j \geq k+1$. Define the mapping*

$$u_1 \oplus u_2 : \bigcup_{p=1}^{k} M_p \longrightarrow N_k$$

as follows:

$$dom(u_1 \oplus u_2) = \bigcup_{p=1}^{k} M_p$$

$$(u_1 \oplus u_2)(s_1 \cup r_1, s_2 \cup r_2) = \overline{u_1}(s_1, s_2) \cup \overline{u_2}(r_1, r_2)$$

for every $(s_1 \cup r_1, s_2 \cup r_2) \in dom(u_1 \oplus u_2)$.

Remark 2.9. *The construction from Definition 2.8 can be applied for the case $u_2 = u_1$ because $S_2 = S_1$ and $S_1 \cap S_2 \neq \emptyset$. For this reason we agree to consider $u_1 \oplus u_1 = u_1$.*

Remark 2.10. *As a conclusion we can relieve the following facts:*

- $u_1 \in R(prod_{S_1})$, $u_2 \in R(prod_{S_2})$
- $T_1 = Cl_{u_1}(T_{01})$, $T_2 = Cl_{u_2}(T_{02})$
- $dom(u_1) \subseteq T_1 \times T_1$, $dom(u_2) \subseteq T_2 \times T_2$
- $dom(u_1 \oplus u_2) = \bigcup_{p=1}^{k} M_p \subseteq N_{k-1} \times N_{k-1} \subseteq (T_1 \Cup T_2) \times (T_1 \Cup T_2)$

- $(u_1 \oplus u_2)(\theta_1, \theta_2) \in N_k \subseteq T_1 \Cup T_2$ *for every* $(\theta_1, \theta_2) \in dom(u_1 \oplus u_2)$

Proposition 2.11. *The mapping* $u_1 \oplus u_2$ *is well defined.*

Proof. We show that for every $(s_1 \cup r_1, s_2 \cup r_2) \in dom(u_1 \oplus u_2)$ we have $(u_1 \oplus u_2)(s_1 \cup r_1, s_2 \cup r_2) \in N_k$. If $(s_1 \cup r_1, s_2 \cup r_2) \in dom(u_1 \oplus u_2) = \bigcup_{p=1}^{k} M_p$ then $(s_1 \cup r_1, s_2 \cup r_2) \in M_j$ for some $j \in \{1, \ldots, k\}$. In this case

$$\overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_j.$$

But $N_j \subseteq N_k$. $\square$

Proposition 2.12. $u_1 \oplus u_2 = u_2 \oplus u_1$

Proof. Consider $N_0 = (f_{01} \sqcup f_{02})(L_{01} \cup L_{02})$ and denote by (N_i, M_i) for $i \geq 1$ the sets defined as in (1) and (2) for $u_1 \oplus u_2$. We denote by (P_i, Q_i) the corresponding sets for $u_2 \oplus u_1$:

$$\begin{cases} P_1 = N_0 \cup \{\mu \mid \exists \rho_1 \cup \omega_1 \in N_0, \exists \rho_2 \cup \omega_2 \in N_0 : \\ \qquad \mu = \overline{u_2}(\rho_1, \rho_2) \cup \overline{u_1}(\omega_1, \omega_2) \neq \emptyset\} \\ P_{k+1} = P_k \cup \{\mu \mid \exists \rho_1 \cup \omega_1 \in P_k, \exists \rho_2 \cup \omega_2 \in P_k : \\ \qquad \mu = \overline{u_2}(\rho_1, \rho_2) \cup \overline{u_1}(\omega_1, \omega_2) \neq \emptyset\} \end{cases}$$

$$\begin{cases} Q_1 = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_0 \times N_0 \mid \overline{u_2}(\rho_1, \rho_2) \cup \overline{u_1}(\omega_1, \omega_2) \in P_1\} \\ Q_{p+1} = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in P_{p-1} \times (P_p \setminus P_{p-1}) \cup (P_p \setminus P_{p-1}) \times P_p : \\ \qquad \overline{u_2}(\rho_1, \rho_2) \cup \overline{u_1}(\omega_1, \omega_2) \in P_{p+1}\} \end{cases}$$

By induction on $i \geq 1$ we can prove that $N_i = P_i$. Consider the sets $Z_1 = N_1 \setminus N_0$ and $W_1 = P_1 \setminus N_0$. Suppose that $(\theta_1, \theta_2) \in Z_1$. There are $\rho_1, \rho_2, \omega_1, \omega_2 \in N_0$ such that $\theta_1 = \rho_1 \cup \omega_1$ and $\theta_2 = \rho_2 \cup \omega_2$ and $(\theta_1, \theta_2) = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset$. Obviously $\theta_1 = \omega_1 \cup \rho_1$ and $\theta_2 = \omega_2 \cup \rho_2$ and $(\theta_1, \theta_2) = \overline{u_2}(\omega_1, \omega_2) \cup \overline{u_1}(\rho_1, \rho_2) \neq \emptyset$. It follows that $Z_1 \subseteq W_1$.

Similarly we have $W_1 \subseteq Z_1$. As a consequence we have $Z_1 = W_1$ and $N_1 = P_1$. Suppose that $N_k = P_k$. Take $(\theta_1, \theta_2) \in N_{k+1}$. If $(\theta_1, \theta_2) \in N_k$ then $(\theta_1, \theta_2) \in P_k$ by the inductive assumption. In this case we have $(\theta_1, \theta_2) \in P_{k+1}$. It remains to consider the case $(\theta_1, \theta_2) \in N_{k+1} \setminus N_k$. There are $\rho_1, \rho_2, \omega_1, \omega_2$ such that $(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_k \times N_k$ such that $(\theta_1, \theta_2) = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset$. We have $(\theta_1, \theta_2) = \overline{u_2}(\omega_1, \omega_2) \cup \overline{u_1}(\rho_1, \rho_2)$. By the inductive assumption we have

$\omega_1 \cup \rho_1 \in P_k$ and $\omega_2 \cup \rho_2 \in P_k$. Thus $(\theta_1, \theta_2) \in P_{k+1} \setminus P_k$. This property shows that $N_{k+1} \subseteq P_{k+1}$. The converse implication is proved in a similar manner.

Based on the fact that $N_i = P_i$ for every $i \geq 1$, it is easy to show by induction on $k \geq 1$ that $M_k = Q_k$. First we have $M_1 = Q_1$ because $N_1 = P_1$. Suppose that $M_i = P_i$ for every $i \in \{1, \ldots, k\}$ and we verify that $M_{k+1} \subseteq P_{k+1}$ and $P_{k+1} \subseteq M_{k+1}$.
Consider $(\theta_1, \theta_2) \in M_{k+1}$. If $(\theta_1, \theta_2) \in M_k$ then by the inductive assumption we have $(\theta_1, \theta_2) \in P_k$. Suppose that $(\theta_1, \theta_2) \in M_{k+1} \setminus M_k$. There are $\rho_1, \rho_2, \omega_1, \omega_2$ such that $(\theta_1, \theta_2) = (\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_{k-1} \times (N_k \setminus N_{k-1}) \cup (N_k \setminus N_{k-1}) \times N_{k-1} \cup (N_k \setminus N_{k-1}) \times (N_k \setminus N_{k-1})$ and $\overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_{k+1}$. By the inductive assumption we obtain

$$(\omega_1 \cup \rho_1, \omega_2 \cup \rho_2) \in P_{k-1} \times (P_k \setminus P_{k-1}) \cup (P_k \setminus P_{k-1}) \times P_{k-1} \cup (P_k \setminus P_{k-1}) \times (P_k \setminus P_{k-1})$$

and

$$\overline{u_2}(\omega_1, \omega_2) \cup \overline{u_1}(\rho_1, \rho_2) \in P_{k+1}.$$

It follows that $(\theta_1, \theta_2) \in Q_{k+1} \setminus Q_k$. The inclusion $P_{k+1} \subseteq M_{k+1}$ is proved in a similar manner.

It follows that

$$dom(u_1 \oplus u_2) = \bigcup_{k \geq 1} M_k = \bigcup_{k \geq 1} Q_k = dom(u_2 \oplus u_1).$$

□

Proposition 2.13. *Consider* $G_1 = (S_1, L_{01}, T_{01}, f_{01}) \in \mathcal{L}_{lg}$, $G_2 = (S_2, L_{02}, T_{02}, f_{02}) \in \mathcal{L}_{lg}$, $\mathcal{G}_1 = (G_1, L_1, T_1, u_1, f^1) \in \mathcal{L}_{sg}$ *and* $\mathcal{G}_2 = (G_2, L_2, T_2, u_2, f^2) \in \mathcal{L}_{sg}$. *If*

$$N_0 = (f_{01} \sqcup f_{02})(L_{01} \cup L_{02})$$

then $Cl_{u_1 \oplus u_2}(N_0) = N_{k_0}$, *where* N_{k_0} *is given by Proposition 2.3.*

Proof. Consider the number k_0 given by Proposition 2.3. In order to obtain $Cl_{u_1 \oplus u_2}(N_0)$ we compute the sequence $\{R_n\}_{n \geq 0}$ defined as follows:

$$\begin{cases} R_0 = N_0 \\ R_{n+1} = R_n \cup \{\theta \mid \exists (\theta_1, \theta_2) \in (R_n \times R_n) \cap dom(u_1 \oplus u_2) : \\ \qquad\qquad \theta = (u_1 \oplus u_2)(\theta_1, \theta_2)\} \end{cases} \tag{4}$$

We verify by induction on $i \geq 0$ that $R_i = N_i$. For $i = 0$ we have $R_0 = N_0$, therefore this property is true for $i = 0$. Suppose the $R_n = N_n$ and we prove that $R_{n+1} = N_{n+1}$. If $\theta \in R_{n+1}$ then we consider the cases $\theta \in R_n$ and $\theta \in R_{n+1} \setminus R_n$. If we have the first case then $\theta \in N_n \subseteq N_{n+1}$. Suppose that we have the second case. There are $(\theta_1, \theta_2) \in (R_n \times R_n) \cap dom(u_1 \oplus u_2)$ such that $\theta = (u_1 \oplus u_2)(\theta_1, \theta_2)$. But $R_n = N_n$ and $\theta = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2)$, therefore (4) can be written as follows:

$$\begin{cases} R_0 = N_0 \\ R_{n+1} = N_n \cup \{\theta \mid \exists(\theta_1, \theta_2) \in (N_n \times N_n) \cap dom(u_1 \oplus u_2) : \\ \qquad\qquad \theta = (u_1 \oplus u_2)(\theta_1, \theta_2)\} \end{cases}$$

therefore $R_{n+1} = N_{n+1}$. $\square$

In order to relieve these aspects we take the following example.

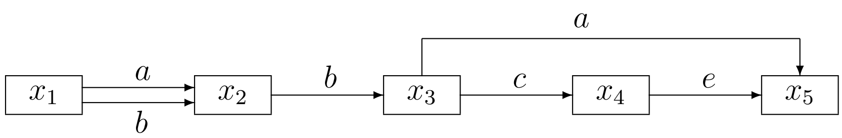

Figure 2: The labeled graph G_1

We consider the labeled graph $G_1 = (S_1, L_{01}, T_{01}, f_{01})$ represented in Figure 2 and defined as follows:

- $S_1 = \{x_1, x_2, x_3, x_4, x_5\}$;
- $L_{01} = \{a, b, c, e\}$;
- $f_{01}(a) = \{(x_1, x_2), (x_3, x_5)\} = \rho_1$; $f_{01}(b) = \{(x_1, x_2), (x_2, x_3)\} = \rho_2$; $f_{01}(c) = \{(x_3, x_4)\} = \rho_3$; $f_{01}(e) = \{(x_4, x_5)\} = \rho_4$;
- $T_{01} = \{\rho_1, \rho_2, \rho_3, \rho_4\}$

We consider the mapping $u_1 \in R(prod_{S_1})$ defined as follows:

$u_1(\rho_1, \rho_2) = \rho_5 = \{(x_1, x_3)\}$; $u_1(\rho_2, \rho_2) = \rho_5$; $u_1(\rho_2, \rho_3) = \rho_6 = \{(x_2, x_4)\}$; $u_1(\rho_1, \rho_6) = \rho_7 = \{(x_1, x_4)\}$; $u_1(\rho_5, \rho_3) = \rho_7$;

This mapping is shortly described in Table 1. It follows that

$$dom(u_1) = \{(\rho_1, \rho_2), (\rho_2, \rho_2), (\rho_2, \rho_3), (\rho_1, \rho_6), (\rho_5, \rho_3)\}$$

It is not difficult to observe that

$$T_1 = Cl_{u_1}(T_{01}) = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7\}$$

Table 1: The mapping u_1

u_1	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
ρ_1		ρ_5				ρ_7	
ρ_2		ρ_5	ρ_6				
ρ_3							
ρ_4							
ρ_5			ρ_7				
ρ_6							
ρ_7							

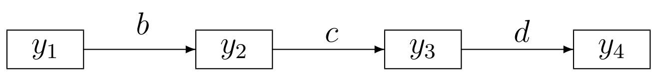

Figure 3: The labeled graph G_2

Let us consider the labeled graph $G_2 = (S_2, L_{02}, T_{02}, f_{02})$ represented in Figure 3 and defined as follows:

- $S_2 = \{y_1, y_2, y_3, y_4\}$;
- $L_{02} = \{b, c, d\}$;
- $f_{02}(b) = \{(y_1, y_2)\} = \omega_1$; $f_{02}(c) = \{(y_2, y_3)\} = \omega_2$; $f_{02}(d) = \{(y_3, y_4)\} = \omega_3$;
- $T_{02} = \{\omega_1, \omega_2, \omega_3\}$

We consider the mapping $u_2 \in R(prod_{S_2})$ defined as follows:

$u_2(\omega_1, \omega_2) = \omega_4 = \{(y_1, y_3)\}$; $u_2(\omega_2, \omega_3) = \omega_5 = \{(y_2, y_4)\}$;
$u_2(\omega_1, \omega_5) = \omega_6 = \{(y_1, y_4)\}$; $u_2(\omega_4, \omega_3) = \omega_6$;

Table 2: The mapping u_2

u_2	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
ω_1		ω_4			ω_6	
ω_2			ω_5			
ω_3						
ω_4			ω_6			
ω_5						
ω_6						

The mapping u_2 is described in Table 2.

We deduce that

$$dom(u_2) = \{(\omega_1, \omega_2), (\omega_2, \omega_3), (\omega_1, \omega_5), (\omega_4, \omega_3)\}$$

We consider now the set $N_0 = (f_{01} \sqcup f_{02})(L_{01} \cup L_{02})$. We have $L_{01} \cup L_{02} = \{a, b, c, d, e\}$ and taking into account the mappings f_{01} and f_{02} we obtain

$$(f_{01} \sqcup f_{02})(a) = f_{01}(a) = \rho_1;\ (f_{01} \sqcup f_{02})(b) = f_{01}(b) \cup f_{02}(b) = \rho_2 \cup \omega_1;$$

$$(f_{01} \sqcup f_{02})(c) = f_{01}(c) \cup f_{02}(c) = \rho_3 \cup \omega_2;\ (f_{01} \sqcup f_{02})(d) = f_{02}(d) = \omega_3;$$

$$(f_{01} \sqcup f_{02})(e) = f_{01}(e) = \rho_4$$

It follows that

$$N_0 = \{\rho_1, \rho_2 \cup \omega_1, \rho_3 \cup \omega_2, \omega_3, \rho_4\}.$$

Further, the computations can be described as follows:

$$(N_0 \times N_0) \cap dom(u_1 \oplus u_2) = \{(\rho_1, \rho_2 \cup \omega_1), (\rho_2 \cup \omega_1, \rho_2 \cup \omega_1), (\rho_2 \cup \omega_1, \rho_3 \cup \omega_2),$$

$$(\rho_3 \cup \omega_2, \omega_3)\}$$

$$N_1 = N_0 \cup \{\rho_5, \rho_6 \cup \omega_4, \omega_5\}$$

$$N_2 = N_1 \cup \{\rho_7, \omega_6\};\ N_3 = N_2$$

$$M_1 = \{(\rho_1, \rho_2 \cup \omega_1), (\rho_2 \cup \omega_1, \rho_2 \cup \omega_1), (\rho_2 \cup \omega_1, \rho_3 \cup \omega_2), (\rho_3 \cup \omega_2, \omega_3)\}$$

$$M_2 = \{(\rho_1, \rho_6 \cup \omega_4), (\rho_2 \cup \omega_1, \omega_5), (\rho_5, \rho_3 \cup \omega_2), (\rho_6 \cup \omega_4, \omega_3)\}$$

$M_3 = \emptyset$

It follows that

$$Cl_{u_1 \oplus u_2} = \{\rho_1, \rho_2 \cup \omega_1, \rho_3 \cup \omega_2, \omega_3, \rho_4, \rho_5, \rho_6 \cup \omega_4, \omega_5, \rho_7, \omega_6\}$$

We obtain the mapping $u_1 \oplus u_2$ from Table 3.

Table 3: The mapping $u_1 \oplus u_2$

$u_1 \oplus u_2$	ρ_1	$\rho_2 \cup \omega_1$	$\rho_3 \cup \omega_2$	ω_3	ρ_4	ρ_5	$\rho_6 \cup \omega_4$	ω_5	ρ_7	ω_6
ρ_1		ρ_5					ρ_7			
$\rho_2 \cup \omega_1$		ρ_5	$\rho_6 \cup \omega_4$					ω_6		
$\rho_3 \cup \omega_2$				ω_5						
ω_3										
ρ_4										
ρ_5			ρ_7							
$\rho_6 \cup \omega_4$				ω_6						
ω_5										
ρ_7										
ω_6										

The next proposition proves the associativity of the operation $\oplus$. First we need several auxiliary results. We mention that we use the following notations and results:

- $G_i = (S_i, L_{0i}, T_{0i}, f_{0i}) \in \mathcal{L}_{lg}$ for $i = 1, 2, 3$
- $\mathcal{G}_i = (G_i, L_i, T_i, u_i, f^i) \in \mathcal{L}_{sg}$ for $i = 1, 2, 3$
- There is k_0 such that $dom(u_1 \oplus u_2) = \bigcup_{p=1}^{k_0} M_p$, where

$$\begin{cases} N_0 = (f_{01} \sqcup f_{02})(L_{01} \cup L_{02}) \\ N_{k+1} = N_k \cup \{\mu \in T_1^{k+1} \uplus T_2^{k+1} \mid \exists \rho_1, \rho_2 \in T_1^k, \exists \omega_1, \omega_2 \in T_2^k : \\ \quad \rho_1 \cup \omega_1 \in N_k, \rho_2 \cup \omega_2 \in N_k; \mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \neq \emptyset\} \end{cases} \tag{5}$$

$$\begin{cases} M_1 = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_0 \times N_0 \mid \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_1\} \\ M_{p+1} = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in N_{p-1} \times (N_p \setminus N_{p-1}) \cup (N_p \setminus N_{p-1}) \times N_p \mid \\ \quad \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\omega_1, \omega_2) \in N_{p+1}\} \end{cases}$$

where $T_1 = Cl_{u_1}(T_{01})$ and $T_2 = Cl_{u_2}(T_{02})$. We denote by $\{T_1^k\}_{k\geq 0}$ and $\{T_2^k\}_{k\geq 0}$ the sequences that give T_1 and T_2 respectively.

Denote $L_{12} = L_{01} \cup L_{02}$ and $g_{12} = f_{01} \sqcup f_{02} : L_{12} \longrightarrow T_{01} \Cup T_{02}$. From (5) we observe that

$$Cl_{u_1 \oplus u_2}(N_0) = N_{k_0}$$

- We consider the following sequences of sets:

$$\begin{cases} P_0 = (g_{12} \sqcup f_{03})(L_{12} \cup L_{03}) \\ P_{k+1} = P_k \cup \{\mu \in N_{k+1} \Cup T_3^{k+1} \mid \exists \rho_1, \rho_2 \in N_k, \exists \omega_1, \omega_2 \in T_3^k : \\ \rho_1 \cup \omega_1 \in P_k, \rho_2 \cup \omega_2 \in P_k; \mu = \overline{u_1 \oplus u_2}(\rho_1, \rho_2) \cup \overline{u_3}(\omega_1, \omega_2) \neq \emptyset\} \end{cases} \tag{6}$$

where $T_3 = Cl_{u_3}(T_{03})$. We denote by $\{T_3^k\}_{k\geq 0}$ the sequence of sets that are used to obtain T_3.

$$\begin{cases} R_1 = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in P_0 \times P_0 \mid \overline{u_1 \oplus u_2}(\rho_1, \rho_2) \cup \overline{u_3}(\omega_1, \omega_2) \in P_1\} \\ R_{p+1} = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in P_{p-1} \times (P_p \setminus P_{p-1}) \cup (P_p \setminus P_{p-1}) \times P_p \mid \\ \qquad \overline{u_1 \oplus u_2}(\rho_1, \rho_2) \cup \overline{u_3}(\omega_1, \omega_2) \in P_{p+1}\} \end{cases}$$

There is m_0 such that $P_{m_0} = P_{m_0+1}$ and $dom((u_1 \oplus u_2) \oplus u_3) = \bigcup_{k=1}^{m_0} R_k$. We remark that

$$P_{m_0} = Cl_{(u_1 \oplus u_2) \oplus u_3}(P_0)$$

- There is s_0 such that $S_{s_0} = S_{s_0+1}$ and $dom(u_2 \oplus u_3) = \bigcup_{p=1}^{s_0} Q_p$, where

$$\begin{cases} S_0 = (f_{02} \sqcup f_{03})(L_{02} \cup L_{03}) \\ S_{k+1} = S_k \cup \{\mu \in T_2^{k+1} \Cup T_3^{k+1} \mid \exists \rho_1, \rho_2 \in T_2^k, \exists \omega_1, \omega_2 \in T_3^k : \\ \qquad \rho_1 \cup \omega_1 \in S_k, \rho_2 \cup \omega_2 \in S_k; \mu = \overline{u_2}(\rho_1, \rho_2) \cup \overline{u_3}(\omega_1, \omega_2) \neq \emptyset\} \end{cases} \tag{7}$$

$$\begin{cases} Q_1 = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in S_0 \times S_0 \mid \overline{u_2}(\rho_1, \rho_2) \cup \overline{u_3}(\omega_1, \omega_2) \in S_1\} \\ Q_{p+1} = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in S_{p-1} \times (S_p \setminus S_{p-1}) \cup (S_p \setminus S_{p-1}) \times S_p \mid \\ \qquad \overline{u_2}(\rho_1, \rho_2) \cup \overline{u_3}(\omega_1, \omega_2) \in S_{p+1}\} \end{cases}$$

Denote $L_{23} = L_{02} \cup L_{03}$ and $g_{23} = f_{02} \sqcup f_{03} : L_{23} \longrightarrow T_{02} \Cup T_{03}$. From (7) we observe that

$$Cl_{u_2 \oplus u_3}(S_0) = S_{s_0}$$

- We consider the following sequences of sets

$$\begin{cases} U_0 = (f_{01} \sqcup g_{23})(L_{01} \cup L_{23}) \\ U_{k+1} = U_k \cup \{\mu \in T_1^{k+1} \uplus S_{k+1} \mid \exists \rho_1, \rho_2 \in T_1^k, \exists \omega_1, \omega_2 \in S_k : \\ \rho_1 \cup \omega_1 \in U_k, \rho_2 \cup \omega_2 \in U_k; \mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2 \oplus u_3}(\omega_1, \omega_2) \neq \emptyset\} \end{cases} \tag{8}$$

$$\begin{cases} V_1 = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in U_0 \times U_0 \mid \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2 \oplus u_3}(\omega_1, \omega_2) \in U_1\} \\ V_{p+1} = \{(\rho_1 \cup \omega_1, \rho_2 \cup \omega_2) \in U_{p-1} \times (U_p \setminus U_{p-1}) \cup (U_p \setminus U_{p-1}) \times U_p \mid \\ \qquad \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2 \oplus u_3}(\omega_1, \omega_2) \in U_{p+1}\} \end{cases}$$

There is j_0 such that $U_{j_0} = U_{j_0+1}$ and $dom(u_1 \oplus (u_2 \oplus u_3)) = \bigcup_{k=1}^{j_0} V_k$. We observe that

$$U_{j_0} = Cl_{u_1 \oplus (u_2 \oplus u_3)}(U_0)$$

Lemma 2.14. *For every $\lambda_1 \cup \gamma_1 \in N_{k_0}$ and $\lambda_2 \cup \gamma_2 \in N_{k_0}$ we have*

$$\overline{u_1 \oplus u_2}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2) \tag{9}$$

Proof. We have

$$\overline{u_1 \oplus u_2}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \begin{cases} u_1 \oplus u_2(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \quad \text{if} \\ \qquad (\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in dom(u_1 \oplus u_2) \\ \emptyset \quad \text{otherwise} \end{cases} \tag{10}$$

We remark that if $(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \notin dom(u_1 \oplus u_2)$ then $\overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2) = \emptyset$. Suppose the contrary, $\overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2) \neq \emptyset$. But $\lambda_1 \cup \gamma_1 \in N_{k_0}$, $\lambda_2 \cup \gamma_2 \in N_{k_0}$ and $N_0 \subseteq N_1 \subseteq N_{k_0} = N_{k_0+1}$. There is $k \leq k_0$ such that $\lambda_1 \cup \gamma_1 \in N_k$ and $\lambda_2 \cup \gamma_2 \in N_k$. If this is the case then $(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in M_k \subseteq dom(u_1 \oplus u_2)$, which is not true. Now, from (10) we obtain

$$\overline{u_1 \oplus u_2}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \begin{cases} u_1 \oplus u_2(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \quad \text{if} \\ \qquad (\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in dom(u_1 \oplus u_2) \\ \overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2) \quad \text{if} \\ \qquad (\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \notin dom(u_1 \oplus u_2) \end{cases} \tag{11}$$

But $u_1 \oplus u_2(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2)$ if $(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in dom(u_1 \oplus u_2)$ and thus from (11) we obtain (9). $\square$

Lemma 2.15. *For every* $\lambda_1 \cup \gamma_1 \in S_{s_0}$, $\lambda_2 \cup \gamma_2 \in S_{s_0}$ *we have*

$$\overline{u_2 \oplus u_3}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \overline{u_2}(\lambda_1, \lambda_2) \cup \overline{u_3}(\gamma_1, \gamma_2) \tag{12}$$

Proof. Directly from the definition of $\overline{u_2 \oplus u_3}$ we obtain

$$\overline{u_2 \oplus u_3}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \begin{cases} u_2 \oplus u_3(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \quad \text{if} \\ \qquad (\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in dom(u_2 \oplus u_3) \\ \emptyset \quad \text{otherwise} \end{cases} \tag{13}$$

We remark that if $(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \notin dom(u_2 \oplus u_3)$ then $\overline{u_2}(\lambda_1, \lambda_2) \cup \overline{u_3}(\gamma_1, \gamma_2) = \emptyset$. Really, let us suppose the contrary, that $\overline{u_2}(\lambda_1, \lambda_2) \cup \overline{u_3}(\gamma_1, \gamma_2) \neq \emptyset$. But $S_0 \subset S_1 \subset S_{s_0} = S_{s_0+1}$ and $\lambda_1 \cup \gamma_1 \in S_{s_0}$, $\lambda_2 \cup \gamma_2 \in S_{s_0}$. There is $s \leq s_0$ such that $\lambda_1 \cup \gamma_1 \in S_s$ and $\lambda_2 \cup \gamma_2 \in S_s$. It follows that $(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in Q_s \subseteq dom(u_2 \oplus u_3)$, which is not true. Now, from (13) we obtain

$$\overline{u_2 \oplus u_3}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \begin{cases} u_2 \oplus u_3(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \quad \text{if} \\ \qquad (\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in dom(u_2 \oplus u_3) \\ \overline{u_2}(\lambda_1, \lambda_2) \cup \overline{u_3}(\gamma_1, \gamma_2) \quad \text{if} \\ \qquad (\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \notin dom(u_2 \oplus u_3) \end{cases} \tag{14}$$

But $u_2 \oplus u_3(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \overline{u_2}(\lambda_1, \lambda_2) \cup \overline{u_3}(\gamma_1, \gamma_2)$ if $(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \in dom(u_2 \oplus u_3)$ and thus from (14) we obtain (12). $\square$

Proposition 2.16.

$$(u_1 \oplus u_2) \oplus u_3 = u_1 \oplus (u_2 \oplus u_3)$$

Proof. We prove that for every $k \geq 0$ we have

$$P_k = U_k \tag{15}$$

For $k = 0$ the relation (15) is true by [12]. Suppose that (15) is true. We verify that

$$P_{k+1} \subseteq U_{k+1} \tag{16}$$

Consider $\mu \in P_{k+1}$. We have the following two cases:

1) If $\mu \in P_k$ then $\mu \in U_k$, therefore in this case $\mu \in U_{k+1}$ and (16) is true.

2) Suppose that $\mu \in P_{k+1} \setminus P_k$. There are $\theta_1 \cup \omega_1 \in P_k$ and $\theta_2 \cup \omega_2 \in P_k$ such that

$$\mu = \overline{u_1 \oplus u_2}(\theta_1, \theta_2) \cup \overline{u_3}(\omega_1, \omega_2) \tag{17}$$

From (6) we have $\theta_1, \theta_2 \in N_k$ and $\omega_1, \omega_2 \in T_3^k$. But $N_k \subseteq T_1^k \Cup T_2^k$, therefore there are $\lambda_1, \lambda_2 \in T_1^k \cup \{\emptyset\}$, $\gamma_1, \gamma_2 \in T_2^k \cup \{\emptyset\}$ such that

$$\theta_1 = \lambda_1 \cup \gamma_1,\ \ \theta_2 = \lambda_2 \cup \gamma_2$$

As a consequence we have $\theta_1 \cup \omega_1 = \lambda_1 \cup \gamma_1 \cup \omega_1$. But $\theta_1 \cup \omega_1 \in P_k$ and $P_k \subseteq N_k \cup T_3^k$. It follows that $\lambda_1 \cup \gamma_1 \cup \omega_1 \in P_k$. But $\omega_1 \in T_3^k$ and thus we obtain $\lambda_1 \cup \gamma_1 \in N_k$. Similarly we have $\lambda_2 \cup \gamma_2 \in N_k$. Applying Lemma 2.14 we obtain

$$\overline{u_1 \oplus u_2}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2) \tag{18}$$

From (17) and (18) we obtain

$$\mu = \overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2) \cup \overline{u_3}(\omega_1, \omega_2)$$

We come back to (17). We have $\theta_1 = \lambda_1 \cup \gamma_1$, $\theta_1 \cup \omega_1 \in P_k$, $\lambda_1 \cup \gamma_1 \cup \omega_1 \in P_k = U_k \subseteq T_1^k \Cup S_k$ and $\lambda_1 \in T_1^k$. It follows that $\gamma_1 \cup \omega_1 \in S_k$. Similarly we have $\gamma_2 \cup \omega_2 \in S_k$.
We can apply Lemma 2.15 and obtain

$$\overline{u_2 \oplus u_3}(\gamma_1 \cup \omega_1, \gamma_2 \cup \omega_2) = \overline{u_2}(\gamma_1, \gamma_2) \cup \overline{u_3}(\omega_1, \omega_2)$$

It follows that

$$\mu = \overline{u_1 \oplus u_2}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \cup \overline{u_3}(\omega_1, \omega_2) = \overline{u_1}(\lambda_1, \lambda_2) \cup \overline{u_2}(\gamma_1, \gamma_2) \cup \overline{u_3}(\omega_1, \omega_2) =$$

$$= \overline{u_1}(\lambda_1, \lambda_2) \cup (\overline{u_2 \oplus u_3})(\gamma_1 \cup \omega_1, \gamma_2 \cup \omega_2)$$

Thus $\mu \in U_{k+1} \setminus U_k$, therefore in this case (16) is true.

The converse inclusion

$$U_{k+1} \subseteq P_{k+1} \tag{19}$$

is proved in a similar manner. Consider $\mu \in U_{k+1}$. We have the following two cases:

1) If $\mu \in U_k$ then $\mu \in P_k$, therefore in this case $\mu \in P_{k+1}$ and (19) is true.

2) Suppose that $\mu \in U_{k+1} \setminus U_k$. There are $\rho_1 \cup \omega_1 \in U_k$ and $\rho_2 \cup \omega_2 \in U_k$ such that

$$\mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2 \oplus u_3}(\omega_1, \omega_2) \tag{20}$$

From (8) we have $\rho_1, \rho_2 \in T_1^k$ and $\omega_1, \omega_2 \in S_k$. But $S_k \subseteq T_2^k \Cup T_3^k$, therefore there are $\lambda_1, \lambda_2 \in T_2^k \cup \{\emptyset\}$, $\gamma_1, \gamma_2 \in T_3^k \cup \{\emptyset\}$ such that

$$\omega_1 = \lambda_1 \cup \gamma_1,\ \ \omega_2 = \lambda_2 \cup \gamma_2$$

We have $\rho_1 \cup \omega_1 = \rho_1 \cup \lambda_1 \cup \gamma_1 \in U_k$ and $U_k = P_k$, therefore $\rho_1 \cup \lambda_1 \cup \gamma_1 \in P_k$. But $\gamma_1 \in T_3^k$, $P_k \subseteq N_k \Cup T_3^k$, and $T_i^k \cap T_j^k = \emptyset$ for $i \neq j$. It follows that $\rho_1 \cup \lambda_1 \in N_k$. From (20) we have

$$\mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2 \oplus u_3}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) \tag{21}$$

But $\lambda_1 \cup \gamma_1 \in S_k$ because $\omega_1 \in S_k$ and $\omega_1 = \lambda_1 \cup \gamma_1$. Similarly we have $\lambda_2 \cup \gamma_2 \in S_k$. We can apply Lemma 2.15 and obtain

$$\overline{u_2 \oplus u_3}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \overline{u_2}(\lambda_1, \lambda_2) \cup \overline{u_3}(\gamma_1, \gamma_2) \tag{22}$$

We have $\omega_1 = \lambda_1 \cup \gamma_1$, $\rho_1 \cup \omega_1 = \rho_1 \cup \lambda_1 \cup \gamma_1 \in U_k = P_k \subseteq N_k \Cup T_3^k$ and $\gamma_1 \in T_3^k$. It follows that $\rho_1 \cup \lambda_1 \in N_k$. Similarly we have $\rho_2 \cup \lambda_2 \in N_k$. We can apply Lemma 2.14 and obtain

$$\overline{u_1 \oplus u_2}(\rho_1 \cup \lambda_1, \rho_2 \cup \lambda_2) = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\lambda_1, \lambda_2) \tag{23}$$

From (21), (22) and (23) we obtain

$$\mu = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2 \oplus u_3}(\lambda_1 \cup \gamma_1, \lambda_2 \cup \gamma_2) = \overline{u_1}(\rho_1, \rho_2) \cup \overline{u_2}(\lambda_1, \lambda_2) \cup \overline{u_3}(\gamma_2, \gamma_2) =$$

$$= \overline{u_1 \oplus u_2}(\rho_1 \cup \lambda_1, \rho_2 \cup \lambda_2) \cup \overline{u_3}(\gamma_1, \gamma_2)$$

Thus $\mu \in P_{k+1} \setminus P_k$, therefore in this case (19) is true and finally (15) is true.

It follows that $V_k = R_k$ for every $k \geq 0$. But

$$dom(u_1 \oplus (u_2 \oplus u_3)) = \bigcup_{k \geq 1} V_k$$

$$dom((u_1 \oplus u_2) \oplus u_3) = \bigcup_{k \geq 1} R_k$$

therefore $dom(u_1 \oplus (u_2 \oplus u_3)) = dom((u_1 \oplus u_2) \oplus u_3)$ and the proposition is proved. $\square$

3 Conclusion

In this paper we used the concept of stratified graph, introduced for the first time in [7]. We know by ([9]) that a mapping u can uniquely define a stratified graph $\mathcal{G}$ over a labeled graph G. We used two mappings u_1 and u_2 that define two stratified graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively and we defined the mapping $u_1 \oplus u_2$. This mapping will be used in further research to generate the least upper bound of stratified graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ over the labeled graph $sup\{G_1, G_2\}$ ([12]). We proved a few properties of the operation $\oplus$, including the associativity.

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