Journal of Applied Mathematics & Bioinformatics, vol.2, no.3, 2012, 165-176
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2012

Asymptotic behavior of some population models

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Abstract

We study the asymptotic behavior of the difference equations

$$x_{n+1} = (\alpha x_n + \beta x_{n-1})e^{-x_n},\ n = 0, 1, 2, ... \qquad (1)$$

and

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n},\ n = 0, 1, 2, ... \qquad (2)$$

where $\alpha, \beta \in (0, \infty)$, which are interesting in their own rights, but which may also be viewed as describing some population models.

Mathematics Subject Classification: 39A10
Keywords: Difference equations, asymptotic, positive solution, equilibrium, nonoscillatory solution, converges

1 Introduction

Equation (1) is derived from a two lifestage model where the young nature into adults, and adults produce young. Such systems have been well studied

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Article Info: *Received* : October 6, 2012. *Revised* : November 12, 2012
Published online : December 30, 2012

when the rates of survival, maturation and reproduction are assumed to be contact. However, they may not be contact. Adults may increase the mortality of the young through cannibalism, exclusion from resources, or transmission of disease. They may inhibit the maturation of the young by shading seedlings if they are trees, or by bihavioral means as in some fish population.

In [5,6] the authors studied the asymptotic behavior of some know population models. They established that every solution of the bounded below by positive constants. They also provided sufficient conditions for the global asymptotic stability of all solution of that higher order difference equations.

The study of a nonoscillatory solution of difference equation converging to the positive equilibrium point $\overline{x}$ is extremely useful in the behavior of mathematical models of various biological systems and other application. This is due to the fact that difference equation are appropriate models for discribing situations where the variable is assumed to take only a discrete set of values and they arise frequently in the study of biological models, in the formation and analysis of discrete - time systems, the numerical intergation of differential equation by finite - difference schemes, the study of deterministic chaos, etc. For example El - Metwally [5] investigated the asymptotic behavior of the population model

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},\ n = 0, 1, 2, ...$$

where α is the immigration rate and β is the population growth rate.

In this paper, we study the asymptotic behavior of the difference equations

$$x_{n+1} = (\alpha x_n + \beta x_{n-1}) e^{-x_n},\ n = 0, 1, 2, ... \tag{1}$$

and

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},\ n = 0, 1, 2, ... \tag{2}$$

where $\alpha, \beta \in (0, \infty)$.

2 Main Results

2.1 Asymptotic aproximation of population model (1)

In this section, we study the nonoscillatory solution of the equation (1) converging to the positive equilibrium point.

The equilibrium point equation is $\overline{x} = (\alpha+\beta)\overline{x}e^{-\overline{x}} \Rightarrow \overline{x} = ln(\alpha+\beta)$ with $\alpha+\beta > 1$.

We pose

$f(x_n, x_{n-1}) = (\alpha x_n + \beta.x_{n-1})e^{-x_n}$

$f'_{x_n}\big|_{\overline{x}} = e^{-\overline{x}}(\alpha - \alpha.\overline{x} - \beta.\overline{x}) = (\alpha+\beta)^{-1}\big[\alpha - (\alpha+\beta).ln(\alpha+\beta)\big] = \dfrac{\alpha}{\alpha+\beta} - ln(\alpha+\beta)$

$f'_{x_{n-1}}\big|_{\overline{x}} = \beta.e^{-\overline{x}} = \dfrac{\beta}{\alpha+\beta}$

Note that the linearized equation of Eq (1) about the positive equilibrium point

$$y_{n+1} = \big[\frac{\alpha}{\alpha+\beta} - ln(\alpha+\beta)\big]y_n + \frac{\beta}{\alpha+\beta}y_{n-1}, n = 0, 1, 2, ... \tag{3}$$

The characteristic polynomial associated with Eq (3) is

$$p(t) = t^2 - \big[\frac{\alpha}{\alpha+\beta} - ln(\alpha+\beta)\big]t - \frac{\beta}{\alpha+\beta} = 0$$

Since,

$$p(0) = -\frac{\beta}{\alpha+\beta} < 0,\ p(1) = 1 - \frac{\alpha}{\alpha+\beta} + ln(\alpha+\beta) - \frac{\beta}{\alpha+\beta} = ln(\alpha+\beta) > 0.$$

$$p'(t) = 2t - \big[\frac{\alpha}{\alpha+\beta} - ln(\alpha+\beta)\big] = 2t + ln(\alpha+\beta) - \frac{\alpha}{\alpha+\beta} > 0 \text{ for } t \in (0,1)$$

It follows that for $\alpha+\beta > 1$, there is a unique positive root $t_0 \in (0,1)$ such that $p(t_0) = 0$ and $0 < t_0^2 < t_0 < 1$ such that $p(t_0^2) < p(t_0) = 0$. It means that

$$p(t_0) = t_0^2 - \big[\frac{\alpha}{\alpha+\beta} - ln(\alpha+\beta)\big]t_0 - \frac{\beta}{\alpha+\beta} = 0$$

$$\Leftrightarrow (\alpha+\beta)t_0^2 - [\alpha - (\alpha+\beta).ln(\alpha+\beta)]t_0 - \beta = 0 \tag{4}$$

This fact motivated us to believe that there are solutions of Eq (3) which have the following asymptotics

$$x_n = \overline{x} + a_1 t_0^n + o(t_0^n) \tag{5}$$

where $a_1 \in R$ and t_0 is the above mentioned root of Eq (4) we solve the open problem, showing that such a solution exists, developing Berg's ideas in [1-4] which are based on the asymptotics. The asymptotics for solutions of difference equation have been investigated by L. Berg and S. Stevi'c, see, for example [7-9], [13, 14] and the reference therein. The problem is solved by constructing appropriate sequences y_n and z_n

$$y_n \leq x_n \leq z_n \tag{6}$$

for sufficiently large n. In [1-4] some methods can be found for the construction of these bounds, see, also [13, 14].

From (5) we expect that for $k \geq 2$ such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \overline{x} + a_1 t_0^n + b_1 t_0^{2n} \tag{7}$$

This is proved by developing Berg's ideas in [1-4] which are based on asymptotics. We need the following result in the proof of main theorems. The proof of the following theorem can be found in [13].

Theorem 2.1. *Let $f : I^{k+2} \to I$ be a continuous and nondecreasing function in each argument on the interval $I \subset \mathbb{R}$, and let $\{y_n\}$ and $\{z_n\}$ be sequences with $y_n \leq z_n$ for $n \geq n_0$ and such that*

$$y_{n-k} \leq f(n, y_{n-k+1}, ..., y_{n+1}), f(n, z_{n-k+1}, ..., z_{n+1}) \leq z_{n-k} \ \ for \ \ n > n_0 + k - 1 \tag{8}$$

Then there is a solution of the following difference equation

$$x_{n-k} = f(n, x_{n-k+1}, ..., x_{n+1}) \tag{9}$$

with property (6) for $n \geq n_0$.

Theorem 2.2. *For α, β are positive constants and $\alpha + \beta > 1$ there is a nonoscillatory solution of Eq (1) converging to the positive equilibrium point $\overline{x} = ln(\alpha + \beta)$ as $n \to \infty$.*

Proof. From Eq (1) we can write in the form

$$F(x_{n-1}, x_n, x_{n+1}) = \frac{1}{\beta} x_{n+1} e^{x_n} - \frac{\alpha}{\beta} x_n - x_{n-1} = g(x_n, x_{n+1}) - x_{n-1} \tag{10}$$

$$g'_{x_{n+1}} = \frac{1}{\beta}e^{x_n} > 0$$
$$g'_{x_n} = \frac{1}{\beta}x_{n+1}e^{x_n} - \frac{\alpha}{\beta} > 0 \text{ with } x_{n+1} > \alpha > \frac{\alpha}{e^{x_n}}$$

we expect the solutions of Eq (1) have the asymptotic appropriation (5)

$$\begin{aligned} F &= F(\varphi_{n-1}, \varphi_n, \varphi_{n+1}) = \frac{1}{\beta}e^{\overline{x}+at^n+bt^{2n}}(\overline{x} + at^{n+1} + bt^{2n+2}) \\ &\quad - \frac{\alpha}{\beta}(\overline{x} + at^n + bt^{2n}) - (\overline{x} + at^{n-1} + bt^{2n-2}) = 0, \end{aligned}$$

for $t \in (0, \infty)$.

$e^{\overline{x}+at^n+bt^{2n}} = [\alpha(\overline{x} + at^n + bt^{2n}) + \beta(\overline{x} + at^{n-1} + bt^{2n-2})](\overline{x} + at^{n+1} + bt^{2n+2})^{-1}$

$$\begin{aligned} \overline{x} + at^n + bt^{2n} &= ln\big[(\alpha+\beta)\overline{x} + \alpha at^n + \beta at^{n-1} + \alpha bt^{2n} + \beta bt^{2n-2}\big] \\ &\quad - ln(\overline{x} + at^{n+1} + bt^{2n+2}) \\ &= ln(\alpha+\beta)\overline{x} + ln\Big[1 + \frac{\alpha at^n + \beta at^{n-1} + \alpha bt^{2n} + \beta bt^{2n-2}}{(\alpha+\beta)\overline{x}}\Big] \\ &\quad - ln\overline{x} - ln\Big(1 + \frac{at^{n+1} + bt^{2n+2}}{\overline{x}}\Big) \end{aligned}$$

$$\begin{aligned} at^n + bt^{2n} &= \frac{\alpha at^n + \beta at^{n-1} + \alpha bt^{2n} + \beta bt^{2n-2}}{(\alpha+\beta)\overline{x}} \\ &\quad - \frac{(\alpha at^n + \beta at^{n-1} + \alpha bt^{2n} + \beta bt^{2n-2})^2}{2(\alpha+\beta)^2\overline{x}^2} \\ &\quad + \frac{(\alpha at^n + \beta at^{n-1} + \alpha bt^{2n} + \beta bt^{2n-2})^3}{3(\alpha+\beta)^3\overline{x}^3} - ... \\ &\quad - \frac{at^{n+1} + bt^{2n+2}}{\overline{x}} + \frac{(at^{n+1} + bt^{2n+2})^2}{2\overline{x}^2} - ... \end{aligned}$$

From (11) we have

$$at^n \equiv \Big(\frac{\alpha a}{(\alpha+\beta)\overline{x}} + \frac{\beta a}{(\alpha+\beta)\overline{x}t} - \frac{at}{\overline{x}}\Big)t^n \Leftrightarrow$$
$$at^n \equiv \frac{-a(\alpha+\beta)t^2 + \alpha at + \beta a}{(\alpha+\beta)\overline{x}t}.t^n \Rightarrow$$
$$a = \frac{-a(\alpha+\beta)t^2 + \alpha at + \beta a}{(\alpha+\beta)\overline{x}t} \Leftrightarrow$$
$$(\alpha+\beta)\overline{x}t = -(\alpha+\beta)t^2 + \alpha t + \beta \Leftrightarrow$$
$$(\alpha+\beta)t^2 + [(\alpha+\beta)\overline{x} - \alpha]t - \beta = 0$$

Posing $t = t_0$ in (11) with t_0 above mentioned, we have

$$p(t_0) = (\alpha+\beta)t_0^2 - [\alpha - (\alpha+\beta)ln(\alpha+\beta)]t_0 - \beta = 0.$$

From (11) it follows

$$\begin{aligned}
bt_0^{2n} &= \frac{\alpha b t_0^{2n} + \beta b t_0^{2n-2}}{(\alpha+\beta)\overline{x}} - \frac{\alpha^2 a^2 t_0^{2n}}{2(\alpha+\beta)^2\overline{x}^2} - \frac{\beta^2 a^2 t_0^{2n-2}}{2(\alpha+\beta)^2\overline{x}^2} - \frac{2\alpha\beta a^2 t_0^{2n-1}}{2(\alpha+\beta)^2\overline{x}^2} - \frac{b}{\overline{x}}t_0^{2n+2} \\
&\quad + \frac{a^2}{2\overline{x}^2}t_0^{2n+2} \\
b &\equiv \frac{\alpha b + \beta b t_0^{-2}}{(\alpha+\beta)\overline{x}} - \frac{\alpha^2 a^2}{2(\alpha+\beta)^2\overline{x}^2} - \frac{\beta^2 a^2 t_0^{-2}}{2(\alpha+\beta)^2\overline{x}^2} - \frac{\alpha\beta a^2 t_0^{-1}}{(\alpha+\beta)^2\overline{x}^2} - \frac{b}{\overline{x}}t_0^2 + \frac{a^2}{2\overline{x}^2}t_0^2 \Rightarrow \\
bt_0^2 &= \frac{\alpha b t_0^2 + \beta b}{(\alpha+\beta)\overline{x}} - \frac{\alpha^2 a^2 t_0^2}{2(\alpha+\beta)^2\overline{x}^2} - \frac{\beta^2 a^2}{2(\alpha+\beta)^2\overline{x}^2} - \frac{\alpha\beta a^2 t_0}{(\alpha+\beta)^2\overline{x}^2} - \frac{b}{\overline{x}}t_0^4 + \frac{a^2}{2\overline{x}}t_0^4 \Rightarrow \\
& b\Big[\Big(-1 + \frac{\alpha}{(\alpha+\beta)\overline{x}} - \frac{t_0^2}{\overline{x}}\Big)t_0^2 + \frac{\beta}{(\alpha+\beta)\overline{x}}\Big] \\
&\quad + a^2\Big[\frac{t_0^4}{2\overline{x}^2} - \frac{\alpha^2 t_0^2}{2(\alpha+\beta)^2\overline{x}^2} - \frac{\alpha\beta t_0}{(\alpha+\beta)^2\overline{x}^2} - \frac{\beta^2}{2(\alpha+\beta)^2\overline{x}^2}\Big] = 0 \Rightarrow \\
& \Big[\frac{-(\alpha+\beta)\overline{x} + \alpha}{(\alpha+\beta)\overline{x}}t_0^2 - \frac{t_0^4}{\overline{x}}\frac{\beta}{(\alpha+\beta)\overline{x}}\Big]b + \\
&\quad + \frac{a^2}{2\overline{x}^2(\alpha+\beta)^2}\Big[(\alpha+\beta)^2 t_0^4 - \alpha^2 t_0^2 - 2\alpha\beta t_0 - \beta^2\Big] = 0 \Rightarrow \\
& \big\{-(\alpha+\beta)t_0^4 + [\alpha - (\alpha+\beta)\overline{x}]t_0^2 + \beta\big\}\frac{b}{(\alpha+\beta)\overline{x}} \\
&\quad + \frac{a^2}{2\overline{x}^2(\alpha+\beta)^2}\big[(\alpha+\beta)^2 t_0^4 - \alpha^2 t_0^2 - 2\alpha\beta t_0 - \beta^2\big] = 0 \Leftrightarrow \\
&\quad \frac{p(t_0^2).b}{(\alpha+\beta)\overline{x}} + \frac{a^2}{2\overline{x}^2(\alpha+\beta)^2}\big[(\alpha+\beta)^2 t_0^4 - \alpha^2 t_0^2 - 2\alpha\beta t_0 - \beta^2\big] = 0
\end{aligned}$$

Finally, we have

$$F = \Big\{\frac{p(t_0^2).b}{(\alpha+\beta)\overline{x}} + \frac{a^2}{2\overline{x}^2(\alpha+\beta)^2}\Big[(\alpha+\beta)^2 t_0^4 - \alpha^2 t_0^2 - 2\alpha\beta t_0 - \beta^2\Big]\Big\}t_0^{2n} + o(t_0^{2n})$$

Setting

$$F = (Bb + C)t_0^{2n} + o(t_0^{2n}),$$
$$H_{t_0}(q) = Bq + C = 0 \to q_0 = -\frac{C}{B},$$
$$H'_{t_0}(q) = B = \frac{p(t_0^2)}{(\alpha+\beta)\overline{x}} < 0$$

we obtain that there are $q_1 < q_0$ and $q_2 > q_0$ such that

$$H_{t_0}(q_1) > 0, \quad H_{t_0}(q_2) < 0, \quad q_1 < q_0 < q_2.$$

We assume that $a \neq 0$, if $\widehat{\varphi}_n = \overline{x} + at_0^n + q_0 t_0^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-1}, \widehat{\varphi}_n, \widehat{\varphi}_{n+1}) \sim [q_0 B + C]t_0^{2n} + o(t_0^{2n})$$

With the notation

$$y_n = \overline{x} + at_0^n + q_1 t_0^{2n}, \; z_n = \overline{x} + at_0^n + q_2 t_0^{2n}$$

We get

$$F(y_{n-1}, y_n, y_{n+1}) \sim [q_1 B + C]t_0^{2n},$$
$$F(z_{n-2}, z_{n-1}, z_n) \sim [q_2 B + C]t_0^{2n}.$$

These relations show that inequalities (8) are satisfied for sufficiently large n, where $g = F + x_{n-1}$ and F is given by (10).
Because the function $g(x_{n-1}, x_n, x_{n+1})$ is continuous and nondecreasing on $[\overline{x}, +\infty)^3 \to [\overline{x}, +\infty)$. We easily have $g(\overline{x}, \overline{x}, \overline{x}) = \overline{x}$. We can apply the Theorem (2.1) with $I = [\overline{x}, \infty)$ and see that there is an $n_0 \geq 0$ and a solution of equation (1) with the asymptotics $x_n = \widehat{\varphi}_n + o(t_0^{2n})$, for $n \geq n_0$ where $b = q_0$ in $\widehat{\varphi}_n$. In particular, the solution converges monotonically to the positive equilibrium point for $n \geq n_0$. The proof is complete. $\square$

2.2 Asymptotic aproximation of population model (2)

In final section, we study the nonoscillatory solution of the equation (2) converging to the positive equilibrium point.

The equilibrium point equation is $\overline{x} = \alpha + \beta.\overline{x}e^{-\overline{x}}$

$$\overline{x} - \beta\frac{\overline{x}}{e^{\overline{x}}} = \alpha$$

We pose

$$\begin{aligned}
&f(x_n, x_{n-1}) = \alpha + \beta . x_{n-1} e^{-x_n} \\
&f'_{x_n}\Big|_{\overline{x}} = -\beta . \overline{x} e^{-\overline{x}} = \alpha - \overline{x} \\
&f'_{x_{n-1}}\Big|_{\overline{x}} = \beta . e^{-\overline{x}} = \frac{\overline{x} - \alpha}{\overline{x}}
\end{aligned}$$

Note that the linearized equation of Eq (2) about the positive equilibrium point

$$y_{n+1} = (\alpha - \overline{x}) y_n + \frac{\overline{x} - \alpha}{\overline{x}} y_{n-1}, n = 0, 1, 2, ... \tag{12}$$

The characteristic polynomial associated with Eq (12) is

$$p(t) = \overline{x} t^2 + \overline{x}(\overline{x} - \alpha) t + \alpha - \overline{x} = 0.$$

Since,

$$p(0) = \alpha - \overline{x} < 0, \ p(1) = \overline{x} + \overline{x}(\overline{x} - \alpha) + \alpha - \overline{x} = \overline{x}(\overline{x} - \alpha) + \alpha > 0$$

$$p'(t) = 2\overline{x} t + \overline{x}(\overline{x} - \alpha) > 0, \quad \text{for} \quad t \in (0, 1).$$

There is a unique positive root $t_0 \in (0, 1)$ such that $p(t_0) = 0$ and $0 < t_0^2 < t_0 < 1$ such that

$$p(t_0^2) < p(t_0) = 0.$$

It means that

$$p(t_0) = \overline{x} t_0^2 + \overline{x}(\overline{x} - \alpha) t_0 + \alpha - \overline{x} = 0.$$

This fact motivated us to believe that there are solutions of Eq (12) which have the following asymptotics

$$x_n = \overline{x} + a_1 t_0^n + o(t_0^n) \tag{13}$$

from (13) we expect that for $k \geq 2$ such solutions have the first three members in their asymptotics in the following form

$$\varphi_n = \overline{x} + a_1 t_0^n + b_1 t_0^{2n}.$$

Theorem 2.3. *For α, β are positive constants there is a nonoscillatory solution of Eq (2) converging to the positive equilibrium point that satisfies*

$$\overline{x}\big(1-\frac{\beta}{e^{\overline{x}}}\big)=\alpha$$

as $n\to\infty$.

Proof. From Eq (2) we can write in the form

$$F(x_{n-1},x_n,x_{n+1})=\frac{1}{\beta}.e^{x_n}(x_{n+1}-\alpha)-x_{n-1}=g(x_n,x_{n+1})-x_{n-1} \tag{14}$$

$$g'_{x_{n+1}}=\frac{1}{\beta}e^{x_n}>0$$
$$g'_{x_n}=\frac{1}{\beta}e^{x_n}(x_{n+1}-\alpha)>0$$

we expect the solutions of Eq (2) have the asymptotic appropriation (13)

$$\begin{aligned} F &= F(\varphi_{n-1},\varphi_n,\varphi_{n+1})=\frac{1}{\beta}e^{\overline{x}+at^n+bt^{2n}}(\overline{x}+at^{n+1}+bt^{2n+2}-\alpha)\\ &\quad -(\overline{x}+at^{n-1}+bt^{2n-2})=0, \end{aligned}$$

for $t\in(0,\infty)$
$e^{\overline{x}+at^n+bt^{2n}}=\beta(\overline{x}+at^{n-1}+bt^{2n-2})(\overline{x}+at^{n+1}+bt^{2n+2}-\alpha)^{-1}$

$$\begin{aligned} \overline{x}+at^n+bt^{2n} &= ln\Big[\beta\big(1+\frac{at^{n-1}}{\overline{x}}+\frac{bt^{2n-2}}{\overline{x}}\big)\big(1+\frac{at^{n+1}}{\overline{x}-\alpha}+\frac{bt^{2n+2}}{\overline{x}-\alpha}\big)^{-1}\Big]\\ &= ln\frac{\beta\overline{x}}{\overline{x}-\alpha}\Big(1+\frac{at^{n-1}}{\overline{x}}+\frac{bt^{2n-2}}{\overline{x}}\Big).\\ &\qquad .\Big[1-\frac{at^{n+1}}{\overline{x}-\alpha}-\frac{bt^{2n+2}}{\overline{x}-\alpha}+\Big(\frac{at^{n+1}+bt^{2n+2}}{\overline{x}-\alpha}\Big)^2-...\Big] \end{aligned}$$

$$\begin{aligned} &ln\overline{x}+\frac{at^{n-1}+bt^{2n-2}}{\overline{x}}-\frac{a^2t^{2n-2}+2abt^{3n-3}+...}{2\overline{x}^2}+...\\ &\quad =\overline{x}+at^n+bt^{2n}+ln\frac{\overline{x}-\alpha}{\beta}+\frac{at^{n+1}+bt^{2n+2}}{\overline{x}-\alpha}-\frac{a^2t^{2n+2}}{2(\overline{x}-\alpha)}+... \end{aligned} \tag{15}$$

From (15) we have

$$ln\overline{x}=\overline{x}+ln\frac{\overline{x}-\alpha}{\beta}\ \text{ or }\ (\overline{x}-\alpha)e^{\overline{x}}=\beta\overline{x}$$

This is trust and from (15) we obtain

$$\frac{a}{\overline{x}}t^{n-1} = at^n + \frac{at^{n+1}}{\overline{x}-\alpha}$$
$$\Big(\frac{a}{t\overline{x}} - a - \frac{at}{\overline{x}-\alpha}\Big)t^n = \frac{[\overline{x}t^2 + t\overline{x}(\overline{x}-\alpha) + (\alpha-\overline{x})]}{t\overline{x}(\overline{x}-\alpha)}at^n = \frac{p(t)at^n}{t\overline{x}(\overline{x}-\alpha)}$$

As mentioned earlier exists $t_0 \in (0,1)$ such that $p(t_0) = 0$ and $0 < t_0^2 < t_0 < 1$, $p(t_0^2) < 0$. Posing $t = t_0$, we get

$$\Big(\frac{a}{t_0\overline{x}} - a - \frac{at_0}{\overline{x}-\alpha}\Big)t_0^n = \frac{p(t_0)at_0^n}{t_0\overline{x}(\overline{x}-\alpha)} = 0$$

From (15) it follows $\frac{b}{\overline{x}}t_0^{2n-2} - \frac{a^2}{2\overline{x}^2}t_0^{2n-2} = bt_0^{2n} + \frac{bt_0^{2n+2}}{\overline{x}-\alpha} - \frac{a^2t_0^{2n+2}}{2(\overline{x}-\alpha)^2}$

$$F = \left\{\Big(\frac{b}{\overline{x}} - \frac{a^2}{2\overline{x}^2}\Big)\frac{1}{t_0^2} - b - \Big[\frac{b}{\overline{x}-\alpha} - \frac{a^2}{2(\overline{x}-\alpha)^2}\Big]t_0^{2n}\right\}t_0^{2n} + o(t_0^{2n})$$
$$= \frac{2b(\overline{x}-\alpha)\overline{x}p(t_0^2) - a^2t_0^4[\overline{x}^2t_0^4 - (\overline{x}-\alpha)^2]}{2\overline{x}^2(\overline{x}-\alpha)^2t_0^2}.t_0^{2n} + o(t_0^{2n})$$

Setting $F = (Bb_1 + C)t_0^{2n} + o(t_0^{2n})$

$$H_{t_0}(q) = Bq + C = 0 \to q_0 = -\frac{C}{B}$$
$$H'_{t_0}(q) = B = \frac{2b(\overline{x}-\alpha)\overline{x}p(t_0^2)}{2\overline{x}^2(\overline{x}-\alpha)^2t_0^2} < 0.$$

We obtain that there are $q_1 < q_0$ and $q_2 > q_0$ such that

$$H_{t_0}(q_1) > 0,\ \ H_{t_0}(q_2) < 0,\ q_1 < q_0 < q_2.$$

We assume that $a \neq 0$, if $\widehat{\varphi}_n = \overline{x} + at_0^n + q_0t_0^{2n}$, we obtain

$$F(\widehat{\varphi}_{n-1}, \widehat{\varphi}_n, \widehat{\varphi}_{n+1}) \sim [q_0B + C]t_0^{2n} + o(t_0^{2n})$$

with the notation

$$y_n = \overline{x} + at_0^n + q_1t_0^{2n}, \quad z_n = \overline{x} + at_0^n + q_2t_0^{2n}.$$

We get

$$F(y_{n-1}, y_n, y_{n+1}) \sim [q_1B + C]t_0^{2n},$$
$$F(z_{n-2}, z_{n-1}, z_n) \sim [q_2B + C]t_0^{2n}.$$

These relations show that inequalities (8) are satisfied for sufficiently large n, where $g = F + x_{n-1}$ and F is given by (14).

Because the function $g(x_{n-1}, x_n, x_{n+1})$ is continuous and nondecreasing on

$[\overline{x}, +\infty)^3 \to [\overline{x}, +\infty)$. We easily have $g(\overline{x}, \overline{x}, \overline{x}) = \overline{x}$. We can apply the Theorem (2.1) with $I = [\overline{x}, \infty)$ and see that there is an $n_0 \geq 0$ and a solution of equation (2) with the asymptotics $x_n = \widehat{\varphi}_n + o(t_0^{2n})$, for $n \geq n_0$ where $b = q_0$ in $\widehat{\varphi}_n$. In particular, the solution converges monotonically to the positive equilibrium point for $n \geq n_0$. The proof is complete. $\square$

Acknowledgements. We would like to extend our thanks to the referees for their suggestions which certainly improved the exposition of our paper.

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