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# On the attraction of positive equilibrium point in Solow economic discrete model with Richards population growth

**N.S. Bay**[1]

**Abstract**

This article presents development of Solow model of macroeconomic growth by replacing the original type population growth by a more innovative population growth. Instead continuous time the problem will be studied for the case of discrete time. These changes bring some better properties in terms of quantitative of the obtained models.

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## 1 Introduction

**Economic variables, production function**. In practice the models written by differential or difference equations attract much attention (see

[1] Department of Mathematics, University of Commerce, Hanoi, Vietnam
e-mail: nsbay2001@yahoo.com

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[1],[3],[5],[9]). There have been many such models in different fields of Engineering, Economic, Social, Environment, Health, etc. Solow model (Nobel Prize in Economics in 1986) is a such model. The basic variables in Solow model of macroeconomic growth are the labor force (denoted by $L$), the amount of capital ($K$), the amount of the output product ($Y$), the ratio of capital to labor ($k = K/L$), the ratio of output product to labor ($y = Y/L$). The important parameters are index accumulation ($s$), index dropped capital ($\delta$). In the macroeconomic production function is given as follows

$$Y(t) = F(K(t), L(t)),$$

where $Y(t)$ is the output, $K(t)$ and $L(t)$ are the inputs. In this article we use a product function of the neo-classical type in the general form. This class is the set of all functions satisfying the following conditions

(i) $F(\lambda K, \lambda L) = \lambda F(K, L) \forall \lambda, K, L \in \mathbb{R}^+$ $(CRS)$.

(ii) $F(K, 0) = F(0, L) = 0, \forall K, L \in \mathbb{R}^+$.

(iii) $\dfrac{\partial F}{\partial K} > 0, \dfrac{\partial F}{\partial L} > 0, \dfrac{\partial^2 F}{\partial K^2} < 0, \dfrac{\partial^2 F}{\partial L^2} < 0.$

(iv) $\lim\limits_{K\to 0} \dfrac{\partial F}{\partial K} = \lim\limits_{L\to 0} \dfrac{\partial F}{\partial L} = +\infty; \lim\limits_{K\to +\infty} \dfrac{\partial F}{\partial K} = \lim\limits_{L\to +\infty} \dfrac{\partial F}{\partial L} = 0$ $(Inada)$.

Denoting $f(k) = F(k, 1) = F(\frac{K}{L}, 1)$, we can see that $f(0) = 0, \lim_{k\to 0^+} f'(k) = +\infty$, $\lim_{k\to +\infty} f'(k) = 0$ and $f(.)$ is strictly concave on $\mathbb{R}^+ := [0; +\infty)$.

**The original Solow model.** Solow model analyzes the relationship between the basic elements of each macro-economic terms. It explains very well the trends and the nature of economic growth in certain conditions. The original Solow model uses the Cobb-Douglas technology, ie take the production function as

$$Y(t) = \gamma K^\alpha(t) L^{1-\alpha}(t) \ (\alpha \in (0; 1))$$

and uses the Malthus population growth: $\dot{L}(t) = nL(t)$ ($n > 0$, const).

The original Solow model is built on the basis of several assumptions quite close to the ideal of economic quantities and remove elements or relationships so complicated. The conditions are: Time is continuous, production platform is simple (not the advancement of technology), production base is closed (pure market, without interference such as international trade, government intervention, the pollution of the environment) and all labor has the work etc. The original Solow model is given by the system of following three equations:

$$\begin{aligned}\dot{L}(t) =& nL(t)\ (n > 0,\ \text{const}, t \in \mathbb{R}^+ := [0; +\infty)),\\ \dot{k}(t) =& s\gamma k^\alpha(t) - (\delta + n)k(t),\\ y(t) =& \gamma k^\alpha(t).\end{aligned}$$

We can see that the original Solow model has a binary nature: Malthus equation of labor growth is not stable because the Lyapunov exponent $\lambda = n > 0$. Volatility equations ratio of capital to labor and ratio of output to labor are asymptotically stable with equilibrium points (see [3]) respectively are:

$$k^* = (\frac{\gamma s}{\delta + n})^{\frac{1}{1-\alpha}};\ y^* = (\frac{\gamma s}{\delta + n})^{\frac{\alpha}{1-\alpha}}.$$

There, instability of the population equation means that when time stretches out indefinitely, the amount of population tends to infinity. This is nonsense because the natural conditions of each economy are being blocked. Bertalanffy (1937, see [3]) refer use symbol $L_\infty$ as a very large positive number characterizing the maximum specific capacity of the environment, which called the environmental carrying capacity. Some models of population growth (Swan (1951), Richards (1959), Schoener (1973) (see [3],[4])) also use this assumption. In addition, other conditions of the Solow model is also very close. It gives us the ability to improve and expand the model. There have been many approaches to expand as the optimal correlation analysis between the parameters (see [4]), added to the effects of external noise (see [6],[7]), edit the elements components of the system (see [1],[3]). In this article, we improve the final oriented model just mentioned, namely labor volatility function changes. The aim of this work is to replace the denominator of the basic economic variables $k, y$ is a more appropriate expression. The determination of the values of the parameters $\delta, s, L^*, L_\infty, \nu$ for the models in a proper sense (best) is not belongs to the scope of this paper and we do not present it here. In this paper, these parameters are considered the given constants.

# 2 Main results

In this section instead for continuous time we will consider the problem for discrete time $\mathbb{Z}^+ := \{0; 1; 2; ...\}$. This approach on the Solow model has been used by Brida and Pareyra in [2]. However, studies of Brida and Pareyra are based on the assumption that rate monotonic growth $n_t$ reduced to 0.We will adjust a little on the function equation Richards of population growth (replate $L_\infty$ by $L^*, L^* \in (0; L_\infty)$), which will lead to three situations: $n_t \equiv 0$, $n_t \uparrow 0$ or $n_t \downarrow 0$. Sufficient condition for monotonic variation of $L_t$ is also given and used. Our investigations will be done with constant growth rate $n_t = n$, then use the dual inequalities and "principle clamping" and move the limit as $t \to +\infty$ to obtain the results for variable growth rate $n_t$.

The experimental research on the nature of the difference equations in general more difficult, fewer tools than the differential equations in the same forms (see [2],[5],[9]). Therefore, the transfer of the results of the continuous model to the discrete time may be less than perfect and often done by other techniques apart. As in the original model, we consider labor accounts for a fixed proportion of the population. This allows us to use the same symbol represented them $L$.

**Richards population growth.** Original Solow model uses Malthus population growth. As known, with this type of growth, when $t \to +\infty$ will lead to $L(t) \to \infty$. It is a major limitation of the model. We will use Bentalanffy's assumption that there is a positive constant $L_\infty$ as the upper bound of the amount of labor $L$. Thus, we are only interested in the case $0 < L < L_\infty$. Every economic choices for itself a value of $L^* \in (0; L_\infty)$ is considered to be ideal, depending on their conditions in all aspects. It would be better if this value $L^*$ is sustained on prolonged period or is asymptotically stable. The following "Re-Richards population growth" is a good variant according to this criterion:

$$\dot{L}(t) = rL(t)[1 - (\frac{L(t)}{L^*})^\nu]\,(\nu \in (0; 1)).$$

For simplicity we call this population growth also is the Richards growth. The advantage of Richards growth against Malthus growth has is obvious: The curve of Malthus growth has exponential behavior on the all semi-axis $\mathbb{R}^+$. While, we can see in the later, the curve of Richards growth has exponential behavior in the first period and has quasi constant behavior when the time tends to infinity. Thus, when $t \to +\infty$ labor force $L(t)$ does not increase to infinity as the old model, but only gradually to the positive equilibrium value $L^*$.

Necessary to add that some other type of population growth, for example

Logistic growth (see [1],[2],[3],[4]) have also the properties similar to those of Richards growth but the differences between their most is: hit of the inflection point of the curve Logistic always equal $L^*/2$ for any initial value $0 < L_0 < L^*$, while hit of the inflection point of the Richards curve depends on parameter $\nu$. This allows us to change it by changing the value of the parameter $\nu$. Recalling that, the inflection point is where the growth curve varies from "quasi exponential" behavior to "quasi constant" behavior.

Discretizing the Richards growth equation on $\mathbb{Z}^+$, we have

$$L_{t+1} = L_t + rL_t[1 - (\frac{L_t}{L^*})^\nu]$$

Finding the solutions of this equation is difficult. We will ignore it and direct evaluation experience through special variation of the growth rate of labor:

$$n_t = \frac{L_{t+1} - L_t}{L_t}$$

and amount of labor $L_t$ of new population growth.

**Solow discrete model with Richards population growth.** Firstly, we determine difference equations of the model. Discretizing difference equation of capital stock changes, we have:

$$K_{t+1} - K_t = sF(K_t, L_t) - \delta K_t, \tag{1}$$

where $s$ is the cumulative index, $\delta$ is the index of capital depreciation ($s, \delta \in (0;1)$). Using discrete Richards population growth for capital volatility equation, we have

$$\begin{aligned}
\frac{K_{t+1}}{L_t} - \frac{K_t}{L_t} =& \frac{s}{L_t}F(K_t; L_t) - \delta\frac{K_t}{L_t} \\
&\Leftrightarrow (1 + n_t)\frac{K_{t+1}}{L_{t+1}} - \frac{K_t}{L_t} = sF(\frac{K_t}{L_t}; 1) - \delta\frac{K_t}{L_t} \\
&\Leftrightarrow (1 + n_t)k_{t+1} - k_t = sf(k_t) - \delta k_t \\
&\Leftrightarrow k_{t+1} = \frac{s}{1 + n_t}f(k_t) + \frac{1 - \delta}{1 + n_t}k_t.
\end{aligned}$$

Combining them, we get the following system:

$$L_{t+1} = L_t + rL_t[1 - (\frac{L_t}{L^*})^\nu], \tag{2}$$

$$k_{t+1} = \frac{s}{1 + n_t}f(k_t) + \frac{1 - \delta}{1 + n_t}k_t, \tag{3}$$

$$y_t = f(k_t). \tag{4}$$

Some time, to distinguish the original Solow model, we call the model described by the system of three equations (2), (3), (4) by phrase "New Solow model".

Necessary to repeat that the constant $x^*$ is called a equilibrium point of the equation $\Delta x_t = x_{t+1} - x_t = \phi(t, x_t)$ if $\phi(t, x^*) = 0, \forall t \in \mathbb{Z}^+$. Due to this reason, two-phrase "$x^*$ is a root of the equation $\phi(t, x_t) = 0$" and "$x^*$ is a equilibrium point of the equation $\Delta x_t = \phi(t, x_t)$" will be understand the same. We denote now:

$$\Omega = \{X = (L; k; y) | 0 < L < L_\infty; k > 0; y > 0\}.$$

For some constant $T > 0$ will be chosen later, we call $k_t^{n_T}; k_t^n; k_t^{n_t}; k_t = k_t^0 (t \in \mathbb{Z}^+, T \in \mathbb{Z}^+)$, the solutions of corresponding equations:

$$k_{t+1} = \frac{s}{1+n_T} f(k_t) + \frac{1-\delta}{1+n_T} k_t, \tag{5}$$

$$k_{t+1} = \frac{s}{1+n} f(k_t) + \frac{1-\delta}{1+n} k_t, \tag{6}$$

$$k_{t+1} = \frac{s}{1+n_t} f(k_t) + \frac{1-\delta}{1+n_t} k_t,$$

$$k_{t+1} = s f(k_t) + (1-\delta) k_t, \tag{7}$$

We denote by $\hat{k}^{n_T}; \hat{k}^n; \hat{k} = \hat{k}^0$ the positive equilibrium points of corresponding equations (5), (6), (7). To prove the main theorem, we need the following propositions.

**Proposition 2.1.** *For any constant $\lambda > 0$, equation $g(t) = 0$, where $g(t) = f(t) - \lambda t$, has exactly one positive solution.*

*Proof.* From $g(0) = 0, \exists f'(t), \forall t \geq 0$ and $\lim_{t \to 0^+} g'(t) = +\infty$ it deduces that there exists a $t_1 > 0$, such that $g(t_1) > 0$. We have also

$$\lim_{t \to +\infty} g'(t) = \lim_{t \to +\infty} (f'(t) - \lambda) = -\lambda < 0.$$

Therefore, there exists a $t_0 > 0$, such that $g'(t) < -\lambda/2, \forall t \geq t_0$. Thus, for $t \geq t_0$, we have $g(t) = g'(s)(t - t_0) < -\lambda(t - t_0)/2$, where $s \in (t_0; t)$. Let $t \to +\infty$, from the last inequality we get $g(t) \to -\infty$. Noting $g(t_1) > 0$ we can see that equation $g(t) = 0$ has least a root denoted by $t_2$, where $t_2 > t_1$. Equation $g(t) = 0$ has not positive roots because $g(0) = g(t_2) = 0$ and $g(.)$ is a strictly concave function on $\mathbb{R}^+$. □

**Proposition 2.2.** *For arbitrary positive constants $\lambda_1; \lambda_2$, let $k_1; k_2$ respectively be the solutions of equations $f(t) = \lambda_1 t$ and $f(t) = \lambda_2 t$. If $\lambda_1 < \lambda_2$ then $k_1 > k_2$.*

*Proof.* Call $M_1; M_2$ corresponding to the intersections of curve $y = f(t)$ to the lines $y = \lambda_1 t; y = \lambda_2 t$. Because $y = f(t)$ is strictly concave on $[0; +\infty)$, therefore $M_1, M_2$ really exist and unique. Since $f(0) = 0$, therefore $\lambda_1 < \lambda_2$ implies that $M_2$ belongs to curve $\widetilde{OM_1}$ of $y = f(t)$. This shows that $k_1 > k_2$, otherwise it will conflicts with the single value of the function $f(.)$. □

**Proposition 2.3.** *If parameter $r$ is small enough such that $0 < r < \frac{L^*}{L_\infty}$, then for any initial value $L_0 \in (0; L_\infty)$ the rate of Richards growth $n_t = \frac{L_{t+1} - L_t}{L_t}$ or equals to $0$ or monotonously tends to $0$ as $t \to \infty$.*

*Proof.* (i) Case $L_0 = L^*$. For this case we have:

$$L_1 - L_0 = rL_0[1 - (\frac{L_0}{L^*})^\nu] = 0. \text{ Thus } L_1 = L_0.$$

Continuing this process, we have:

$$L_2 = L_1;\ L_3 = L_2;\ ...\ L_{t+1} = L_t, \forall t \in \mathbb{Z}^+.$$

In this case we get $n_t \equiv 0$.
(In this case $L_0$ is the positive equilibrium point of equation (2)).
(ii) Fore the case $0 < L_0 < L^*$, we need to show that:

$$L_t < L_{t+1} < L^*, \forall t \in \mathbb{Z}^+.$$

Indeed, $L_0 < L^* \Rightarrow \dfrac{L_0}{L^*} < 1$. Therefore $L_1 - L_0 = rL_0[1 - (\dfrac{L_0}{L^*})^\nu] > 0$ or $L_1 > L_0$.
We need to show also $L_1 < L^*$. Indeed,

$$L_1 = L_0 + rL_0[1 - (\frac{L_0}{L^*})^\nu].$$

Because $0 < L_0 < L^*$ and $\nu \in (0; 1)$ so:

$$L_1 = L_0 + rL_0[1 - (\frac{L_0}{L^*})^\nu] < L_0 + rL_0[1 - \frac{L_0}{L^*}].$$

We see that:

$$\begin{aligned}
&L_1 < L^* \\
&\Leftrightarrow L_0 + rL_0[1 - \frac{L_0}{L^*}] < L^* \\
&\Leftrightarrow \frac{rL_0}{L^*}(L^* - L_0) < L^* - L_0 \\
&\Leftrightarrow r < \frac{L^*}{L_0}.
\end{aligned}$$

The last inequality holds because $r < \frac{L^*}{L_\infty} < 1$ and $0 < L_0 < L^*$. Thus, $0 < L_0 < L^* \Rightarrow L_0 < L_1 < L^*$
Continuing by the inductive method, we can show that

$$L_0 < L_1 < ... < L_t < L_{t+1} < ... < L^*, \forall t \in \mathbb{Z}^+.$$

Thus, sequence $\{L_t\}$ is monotonically increasing and bounded upper. So it has a limit as $t \to \infty$. Denote this limit by $\bar{L} = \lim_{t\to\infty} L_t$. We show that $\bar{L} = L^*$. Indeed, we have $\lim_{t\to\infty} L_t = \lim_{t\to\infty} L_{t+1} = \bar{L}$. Therefore $0 = \lim_{t\to\infty}(L_{t-1} - L_t) = \lim_{t\to\infty} rL_t[1 - (\frac{L_t}{L^*})^\nu] = r\bar{L}[1 - (\frac{\bar{L}}{L^*})^\nu]$. It means that $\bar{L}$ is a positive equilibrium point of equation (2). From the uniqueness of the positive equilibrium point of the equation (2) we have $\bar{L} = L^*$. In short, if $0 < L_0 < L^*$ then $L_t$ monotonically increases to $L^*$. Finally, we need show that $n_t$ monotonically decreases to 0. Indeed,

$$\begin{aligned}
n_{t+1} < n_t \Leftrightarrow r(1 - (\frac{L_t)}{L^*})^\nu) < r(1 - (\frac{L_t)}{L^*})^\nu) \Leftrightarrow -(\frac{L_t)}{L^*})^\nu < -(\frac{L_{t-1})}{L^*})^\nu \\
\Leftrightarrow (\frac{L_t)}{L^*})^\nu > (\frac{L_{t-1})}{L^*})^\nu \Leftrightarrow (\frac{L_t)}{L^*}) > (\frac{L_{t-1})}{L^*}) \Leftrightarrow L_t > L_{t-1}.
\end{aligned}$$

The last inequality is shown before.
(iii) For the case $L_\infty > L_0 > L^*$, we need to show that:

$$L_t > L_{t+1} > L^*, \forall t \in \mathbb{Z}^+.$$

Indeed, $L_0 > L^* \Rightarrow \frac{L_0}{L^*} > 1$. Therefore $L_1 - L_0 = rL_0[1 - (\frac{L_0}{L^*})^\nu] < 0$ or $L_1 < L_0$.
We need to show also $L_1 > L^*$. Indeed,

$$L_1 = L_0 + rL_0[1 - (\frac{L_0}{L^*})^\nu]$$

Since $L_0 > L^* > 0$ and $\nu \in (0;1)$ therefore $L_1 = L_0 + rL_0[1 - (\frac{L_0}{L^*})^\nu] > L_0 + rL_0[1 - \frac{L_0}{L^*}]$. Then:

$$\begin{aligned}
& L_1 > L^* \\
& \Leftrightarrow L_0 + rL_0[1 - \frac{L_0}{L^*}] > L^* \\
& \Leftrightarrow \frac{rL_0}{L^*}(L^* - L_0) > L^* - L_0 \\
& \Leftrightarrow r < \frac{L^*}{L_0}.
\end{aligned}$$

Last inequality holds because $r < \frac{L^*}{L_\infty}$ and $L_\infty > L^*$. In the short, from $L_\infty > L_0 > L^*$, we have $L_0 > L_1 > L^*$. By inductive method we get that

$$L_0 > L_1 > ... > L_t > L_{t+1} > ... > L^*, \forall t \in \mathbb{Z}^+.$$

Thus, sequence $\{L_t\}$ is monotonically decreasing and bounded under. By the same way above, this sequence has limit $L^*$.
Now we will show that $n_t$ monotonically increases to 0. Indeed,

$$\begin{aligned}
& n_{t+1} > n_t \Leftrightarrow r(1 - (\frac{L_t)}{L^*})^\nu) > r(1 - (\frac{L_t)}{L^*})^\nu) \Leftrightarrow -(\frac{L_t)}{L^*})^\nu > -(\frac{L_{t-1})}{L^*})^\nu \\
& \Leftrightarrow (\frac{L_t)}{L^*})^\nu < (\frac{L_{t-1})}{L^*})^\nu \Leftrightarrow (\frac{L_t)}{L^*}) < (\frac{L_{t-1})}{L^*}) \Leftrightarrow L_t < L_{t-1}.
\end{aligned}$$

This statement was shown above. □

**Remark 2.1.** From this proposition, it is useful to note that the linear Malthus growth is not stable, but adding there a square noise $-rL^2/L^*$ obtained equation becomes asymptotically stable.

**Proposition 2.4.** *For any nonnegative constant $n$ and any $k_0 > 0$ solution $k_t$ beginning from $k_0$ of equation*

$$k_{t+1} = \frac{s}{1+n} f(k_t) + \frac{1-\delta}{1+n} k_t$$

*is being attracted to equlibrium point $\hat{k}^n$ of this equation.*

*Proof.* (i) Case $k_0 = \hat{k}^n$. For this case we have

$$k_1 = \frac{s}{1+n} f(k_0) + \frac{1-\delta}{1+n} k_0 = \frac{s}{1+n} f(\hat{k}^n) + \frac{1-\delta}{1+n} \hat{k}^n = \hat{k}^n.$$

Continuing process, we get:

$$k_0 = k_1 = k_2 = ... = \hat{k}^n.$$

It means that $\hat{k}^n$ is a positive equilibrium point of equation (6).
(ii) For the case $k_0 < \hat{k}^n$. Firstly, we show that if $k_t < \hat{k}^n$ then $k_{t+1} < k_t < \hat{k}^n$. Indeed,

$$\begin{aligned} k_{t+1} &= \frac{s}{1+n} f(k_t) + \frac{1-\delta}{1+n} k_t \\ \Leftrightarrow k_{t+1} - k_t &= \frac{s}{1+n} f(k_t) - \frac{\delta+n}{1+n} k_t \end{aligned}$$

We construct a function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as follows:

$$g(k) = \frac{s}{1+n} f(k) - \frac{\delta+n}{1+n} k$$

$$g(k) = 0 \Leftrightarrow sf(k) = (n+\delta)k = 0 \Leftrightarrow k = \hat{k}^n; k = 0.$$

For $0 < k < \hat{k}^n$, we can write $k$ as a convex combination of two points 0 and $\hat{k}^n$ as $k = (1-\lambda).0 + \lambda\hat{k}^n = \lambda\hat{k}^n$, where $0 < \lambda < 1$. Because $f(.)$ is strictly concave on $\mathbb{R}^+$, we have

$$\begin{aligned} g(k) = g(\lambda\hat{k}^n) &= \frac{s}{1+n} f[(1-\lambda).0 + \lambda\hat{k}^n] - \frac{\delta+n}{1+n}[(1-\lambda).0 + \lambda\hat{k}^n] \\ &= \frac{s}{1+n} f[(1-\lambda).0 + \lambda\hat{k}^n] - \lambda\frac{\delta+n}{1+n}\hat{k}^n \\ &> \lambda\frac{s}{1+n} f(\hat{k}^n) - \lambda\frac{\delta+n}{1+n}\hat{k}^n = 0. \end{aligned}$$

Thus, when $k_t < \hat{k}^n$ we have

$$k_{t+1} - k_t = g(k_t) = \frac{s}{1+n} f(k_t) - \frac{\delta+n}{1+n} k_t > 0 \Leftrightarrow k_{t+1} > k_t.$$

Next, since $k_t < \hat{k}^n$ and $f'(k) > 0$, we have:

$$k_{t+1} = \frac{s}{1+n} f(k_t) + \frac{1-\delta}{1+n} k_t < \frac{s}{1+n} f(\hat{k}^n) + \frac{1-\delta}{1+n} \hat{k}^n = \hat{k}^n.$$

Return to $t = 0$. From assumption $0 < k_0 < \hat{k}^n$ according the above proof, we have

$$k_0 < k_1 < k_2 < ... < k_t < k_{t+1} < ... < \hat{k}^n.$$

Thus, sequence $\{k_t\}$ is monotonically increasing and bounded upper by $\hat{k}^n$. Therefore, it has a limit. Put $\lim_{t\to\infty} k_t = \bar{k}$. We need to show that $\bar{k} = \hat{k}^n$. Indeed, since $\lim_{t\to\infty} k_t = \lim_{t\to\infty} k_{t+1} = \bar{k}$ we have

$$0 = \lim_{t\to\infty}(k_{t+1} - k_t) = \lim_{t\to}(\frac{s}{1+n}f(k_t) - \frac{\delta+n}{1+n}k_t)$$

$$= \frac{s}{1+n}f(\bar{k}) - \frac{\delta+n}{1+n}\bar{k}$$

This means that $\bar{k}$ is also a positive root of $g(k) = 0$. Uniqueness of positive root of this equation implies that $\bar{k} = \hat{k}^n$. Thus, it is shown that if $k_0 < \hat{k}^n$ then $k_t$ monotonically increases to $\hat{k}^n$.
(iii) Fore the case $k_0 > \hat{k}^n$ by the similar ways we can show that sequence $k_t$ monotonically decreases to $\hat{k}^n$. □

**Remark 2.2.** If $n = 0$ equation (6) becomes (7) as follows:

$$k_{t+1} = sf(k_t) + (1-\delta)k_t$$

There we denote it's positive equilibrium point by $\hat{k}^0$ or more simply by $\hat{k}$.

**Proposition 2.5.** *For any integer positive constant $T$, solutions of equations $(3), (5), (7)$ with the same initial value $k_T = k_0 > 0$ at $t = T$ satisfy the following order:*

$$(i) \qquad k_t^0 \geq k_t^{n_t} \geq k_t^{n_T}, \forall t \geq T \quad \textit{for the case} \quad n_t \downarrow 0.$$

$$(ii) \qquad k_t^0 \leq k_t^{n_t} \leq k_t^{n_T}, \forall t \geq T \quad \textit{for the case} \quad n_t \uparrow 0.$$

*Proof.* Firstly, we check the right inequality: $k_t^{n_t} \geq k_t^{n_T}$. Indeed, at $t = T$ this inequality satisfied by assuming the same initial conditions. Suppose that this inequality is true at step $t\,(t \geq T)$. We check for steps $t+1$. Since $\delta \in (0;1)$ and $f'(t) > 0$ so $k_t^{n_t} \geq k_t^{n_T}$ involves $f(k_t^{n_t}) \geq f(k_t^{n_T})$. On the other hand, because $n_t$ monotonically decreases to 0 as $t \to +\infty$ therefore:

$$\begin{aligned} k_{t+1}^{n_t} &= \frac{s}{1+n_t} f(k_t^{n_t}) + \frac{1-\delta}{1+n_t} k_t^{n_t} \\ &\geq \frac{s}{1+n_t} f(k_t^{n_T}) + \frac{1-\delta}{1+n_t} k_t^{n_T} \\ &\geq \frac{s}{1+n_T} f(k_t^{n_T}) + \frac{1-\delta}{1+n_T} k_t^{n_T} = k_{t+1}^{n_T}. \end{aligned}$$

The right inequality is proven. Next we proof the left inequality: $k_t^0 \geq k_t^{n_t}$. Indeed, at $t = T$ this inequality satisfied by assuming the same initial conditions. Suppose that the inequality is true in step $t \, (t \geq T)$. We check for steps $t+1$. Since $\delta \in (0;1)$ and $f'(t) > 0$ so $k_t^0 \geq k_t^{n_t}$ implies $f(k_t^0) \geq f(k_t^{n_t})$. Thus:

$$\begin{aligned} k_{t+1}^{0} &= \frac{s}{1+0} f(k_t^{0}) + \frac{1-\delta}{1+0} k_t^{0} \\ &\geq \frac{s}{1+0} f(k_t^{n_t}) + \frac{1-\delta}{1+0} k_t^{n_t} \\ &\geq \frac{s}{1+n_t} f(k_t^{n_t}) + \frac{1-\delta}{1+n_t} k_t^{n_t} = k_{t+1}^{n_t}. \end{aligned}$$

The double inequality is proved.
(ii) For the case $n_t \uparrow 0$, the proof is similar to (i). □

**Corollary 2.1.** *With the same assumptions as in Proposition 2.4, independently to initial values, we have*

$$\begin{aligned} &(i) \qquad \hat{k}^0 \geq \hat{k}^{n_T} \quad \textit{for the case} \quad n_t \downarrow 0. \\ &(ii) \qquad \hat{k}^0 \leq \hat{k}^{n_T} \quad \textit{for the case} \quad n_t \uparrow 0. \end{aligned}$$

*Proof.* (i) For the same initial conditions at $t = T$, according to Proposition 2.5, we have:

$$k_t \geq k_t^{n_T}, \forall t > T.$$

According Proposition 2.4, the upper limit of all solutions of equation is the same, which is not depend to the initial value $k_0$. Therefore, letting $t \to \infty$, we have

$$\hat{k} \geq \hat{k}^{n_T}$$

(ii) This statement is proved by the similar way. □

**Proposition 2.6.** *The following relation holds:*

$$\lim_{T\to\infty} \hat{k}^{n_T} = \hat{k}$$

*Proof.* (i) It is obvious for the case $n_t \equiv 0$.
(ii) If $n_t \downarrow 0$ then for any positive integer $T$ we have $n_{T+1} < n_T$. Then, according Proposition 2.5 and Corollary 2.1, we have

$$\hat{k}^{n_T} \leq \hat{k}^{n_{T+1}} \leq \hat{k}^0.$$

Thus, sequence $\hat{k}^{n_T}$ is monotonically increasing and bounded upper as $n_T \to 0^+$ or as $T \to \infty$. Therefore, this sequence has a limit as $T \to \infty$. Denote this limit by $\bar{k}$. From above inequality transferring to limits as $T \to \infty$, we have $\bar{k} \leq \hat{k}^0$. We will show that $\bar{k} = \hat{k}^0$. Indeed, since $\hat{k}^{n_T}$ is a positive equilibrium point of (5) therefore

$$sf(\hat{k}^{n_T}) = (\delta + n_T)\hat{k}^{n_T}.$$

Transferring to limits as $T \to +\infty$, we have

$$sf(\bar{k}) = \delta\bar{k}.$$

Thus, $\bar{k}$ is also a positive equilibrium point of (7), therefore $\bar{k} = \hat{k}^0$. Thus, we prove that

$$\lim_{T\to\infty} \hat{k}^{n_T} = \hat{k}^0 = \hat{k}.$$

(iii) Case $n_t \uparrow 0$. For this case, the proof is similar as above.
The proof is similar to the previous parts. □

**Theorem 2.1.** *If own rate $r$ of Richards population growth is small enough so $r < \frac{L^*}{L_\infty}$ then for any initial point $X^0 = (L_0, k_0, y_0) \in \Omega$ the solution $X(t) = (L(t), k(t), y(t))$ begining from $X^0$ of Solow discrete model (2), (3), (4) attracts to $X^* = (L^*, \hat{k}, \hat{y})$, where $\hat{k}$ is the positive equilibrium point of equation (7) and $\hat{y} = f(\hat{k})$.*

*Proof.* (i) For the case $n_t \downarrow 0$.

According to the above propositions, we have that for each positive integer $T$, taking the same initial conditions at $t = T$ for $k_t^0, k_T^{n_t}, k_T^{n_T}$ the following relations hold:

$$k_t^0 \geq k_t^{n_t} \geq k_t^{n_T}, \forall t \geq T.$$

$$k_t^0 \to \hat{k} \text{ as } t \to \infty.$$

$$k_t^{n_T} \to \hat{k}^{n_T} \text{ as } t \to \infty.$$

Transferring to limits as $t \to \infty$, we have

$$\lim_{t\to\infty} k_t^0 \geq \lim_{t\to\infty} k_t^{n_t} \geq \lim_{t\to\infty} k_t^{n_T}.$$

Letting $T \to \infty$, we have

$$\lim_{T\to\infty}\lim_{t\to\infty} k_t^0 \geq \lim_{T\to\infty}\lim_{t\to\infty} k_t^{n_t} \geq \lim_{T\to\infty}\lim_{t\to\infty} k_t^{n_T}.$$

$$\Leftrightarrow \hat{k} \geq \lim_{T\to\infty}\lim_{t\to\infty} k_t^{n_t} \geq \hat{k}.$$

Thus,

$$\lim_{T\to\infty}\lim_{t\to\infty} k_t^{n_t} = \hat{k}.$$

And therefore

$$\lim_{t\to\infty} k_t^{n_t} = \hat{k}.$$

The statement on ratio of product to labor $y$ is directly implied from (4), $f'(t) > 0$ and all above statements on $k$.

(ii) For the case $n_t \uparrow 0$, the proof is similar. Theorem is proved. □

**Remark 2.3.** The proved statements in this theorem are not other as the state $(L^*, \hat{k}, \hat{y})$ of obtained model is asymptotically stable (see [5],[6],[7). This positive equilibrium state is globally attracted from the set $\Omega$.

**Theorem 2.2.** *Suppose that all assumptions of Theorem 2.1 hold and all parameters, production function are same. Then for Solow discrete model with Richards population growth* (2), (3), (4) *at positive equilibrium point the ratio of capital on labor and ratio of product on labor really higher than in Solow discrete model with Malthus population growth.*

*Proof.* Now we denote $k^{**}, y^{**}$ being the positive equilibrium points of according equations in Solow discrete model with Malthus population growth. For the New model: Though the above statements we know that $\hat{k}$ is the positive root of equation

$$sf(k) - \delta k = 0,$$

while, $k^{**}$ is the positive root of equation

$$sf(k) - (\delta + n)k = 0.$$

Since $n > 0$ or $\delta + n > \delta$. Using Proposition 2.2, we get $\hat{k} > k^{**}$. About $y^{**}$, since $y = f(k)$ and $f'(k) > 0, \forall k \in [0; +\infty)$, we have also $\hat{y} > y^{**}$. Theorem is proved. $\square$

# 3 Conclusion

In this paper, the Solow economic model with Richards population growth and with product function in general form has been considered in discrete time. For this proposed model the positive equilibrium point is globally attracted on a some set $\Omega \subset \mathbb{R}^3$ and for this discrete model, at the equilibrium state the basic values are really higher than in the original model.

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# References

[1] E. Accinelli and J.G. Brida, The dynamics of the Ramsey economic growth model with the von Bertalanffy population growth law, *AMS*, **1**(3), (2007), 109–118.

[2] J.G. Brida and J.S. Pareyra, The Solow model in discrete time and decreasing population growth rate, *Economic bulletin*, **3**(4), (2008), 1–14.

[3] J. G. Brida and E. Maldonado, Closed form solutions to a Generalization of the Solow growth model, *AMS*, **1**(40), (2009), 1991–2000.

[4] W.A. Brock and M.S. Taylor,The green Solow model, *NBER Working paper series*, **10557**, (2004).

[5] N.S. Bay and V.N. Phat, Stability analysis of nonlinear retarded difference equations in Banach spaces, *J. Computers and Mathematics with Applications*, **45**, (2003), 951–960.

[6] N.S. Bay, Stability and stabilization of nonlinear time-varying delay systems with non-autonomous kernels, *Advances in Nonlinear Variational Inequalities*, **13**(2), (2010), 59–69.

[7] N.S. Bay, V.N. Phat and N.M. Linh, Further results on $H_\infty$ control of linear non-autonomous systems with mixed time-varying delays, *Optimal Control, Applications and Methods*, **32**, (2011), 545–557.

[8] H.I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.

[9] M. Ferrara and L. Guerrini, The green Solow model with logistic population change, *Proc. 10th Int. Conf. on Math. and Comp. in Business and Economic*, (2009), 191–200.