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A priori estimate and continuous dependence of solutions to mixed boundary value problems for pseudo-parabolic equation

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Abstract

One establishes the a priori energy inequality which guarantees the uniqueness of the solution and shows the continuous dependence of the solutions of the form of the boundary conditions.

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1 Introduction

In a rectangle, $\Omega = (0,l) \times (0,T)$, we study the set of mixed problems with

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integral conditions

$$\frac{\partial u_\alpha}{\partial t} - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u_\alpha}{\partial x \partial t}\right) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u_\alpha}{\partial x}\right) = f(x,t) \tag{1}$$

$$u_\alpha(x,0) = \varphi_\alpha(x), \quad u_\alpha(0,t) = 0, \quad \frac{1}{l-\alpha}\int_\alpha^l u_\alpha(x,t)dx = h(t) \tag{2}$$

Where $0 \leqslant \alpha < l$ and the mixed problem with local conditions

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x \partial t}\right) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t) \tag{3}$$

$$u(x,0) = \varphi(x), \quad u(0,t) = 0, \quad u(l,t) = h(t) \tag{4}$$

It is assumed that the following conditions are satisfied

Condition 1.

For all $(x,t) \in \Omega$, we suppose that

$a_0 \le a(x,t) \le a_1, \quad \frac{\partial a(x,t)}{\partial t} \le a_2, \quad \frac{\partial a(x,t)}{\partial x} \le a_3, \quad \frac{\partial^2 a(x,t)}{\partial x \partial t} \le a_4;$

$1 \le b(x,t) \le b_1, \quad \frac{\partial b(x,t)}{\partial t} \le b_2, \quad \frac{\partial b(x,t)}{\partial x} \le b_3, \quad \frac{\partial^2 b(x,t)}{\partial t^2} \le b_4;$

$\frac{\partial^2 b(x,t)}{\partial x \partial t} \le b_5, \quad \frac{\partial^2 b(x,t)}{\partial x^2} \le b_6, \quad \frac{\partial^3 b(x,t)}{\partial x \partial t^2} \le b_7 \ ;$

$(a_i)_{0\le i\le 4}, \quad (b_k)_{1\le k\le 7}$ *are positive constants.*

Condition 2.

$f \in L_2(\Omega),\ h \in W_2^1(0,T),\ \varphi_\alpha,\ \varphi \in W_2^1(0,T),\ \varphi_\alpha(0) = \varphi(0) = 0,\ \varphi(1) = h(0),$

$\frac{1}{l-\alpha}\int_\alpha^l \varphi_\alpha(x)dx = h(0).$

2 Preliminary Notes

In a rectangle $\Omega = (0,l) \times (0,T)$, consider equation:

$$\mathcal{L}_\lambda u = \frac{\partial u}{\partial t} - \lambda\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x \partial t}\right) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t) \tag{5}$$

with the initial condition

$$u(x,t)|_{t=0} = \varphi(x),\ x \in (0,l), \tag{6}$$

local boundary conditions

$$\begin{cases} u(x,t)|_{x=0} = 0, \text{ (conditions homogeneous to the limits of Dirichlet)} \\ \dfrac{\partial u}{\partial x}(x,t)|_{x=0} = 0, \text{ (conditions homogeneous to the limits of Neumann)} \end{cases} \tag{7}$$

and the homogeneous nonlocal condition

$$\int_{\alpha}^{l} u(x,t)dx = 0, \ 0 \leqslant \alpha < l, t \in (0,T) \tag{8}$$

where the functions,$a(x,t); b(x,t); f(x,t); \varphi(x)$, require the following conditions:

Condition 3.

$$a_0 \leq a(x,t) \leq a_1; \quad \frac{\partial a(x,t)}{\partial t} \leq a_2; \quad \frac{\partial a(x,t)}{\partial x} \leq a_3; \quad \frac{\partial^2 a(x,t)}{\partial x \partial t} \leq a_4$$

$$1 \leq b(x,t) \leq b_1; \quad \frac{\partial b(x,t)}{\partial t} \leq b_2; \quad \frac{\partial b(x,t)}{\partial x} \leq b_3; \quad \frac{\partial^2 b(x,t)}{\partial x^2} \leq b_4$$

$$\frac{\partial^2 b(x,t)}{\partial x \partial t} \leq b_5; \frac{\partial^2 b(x,t)}{\partial t^2} \leq b_6; \quad \frac{\partial^3 b(x,t)}{\partial x^2 \partial t} \leq b_7$$

with the $(a_i)_{o \leq i \leq 4}$; $(b_k)_{1 \leq k \leq 7}$ *and* λ *positive real constants.*

Condition 4.
$\varphi(0) = 0$ *et* $\int_{\alpha}^{l} \varphi(x)dx = 0$
When the homogeneous condition at the limits of Dirichet is used .
$\varphi'(0) = 0$ *et* $\int_{\alpha}^{l} \varphi(x)dx = 0$
When Neumann's homogeneous condition at the limits is used.

2.1 Basic lemma

The solution of the problem (5) to (8) can be considered as the solution of the operational equation:

$$\mathcal{L}_{\lambda} u = \mathcal{F} \equiv (f, \varphi) \tag{9}$$

Where the operator $\mathcal{L}_\lambda$ defined by:

$$\mathcal{L}_\lambda : E_\lambda \subseteq L^2(\Omega) \longrightarrow F$$
$$u \longmapsto \mathcal{L}_\lambda u = (\mathcal{L}_\lambda u, \varphi)$$

• has as its domain:
$D(\mathcal{L}_\lambda) = \{u \in L^2(\Omega)/ \frac{\partial u}{\partial t}; \quad \frac{\partial^2 u}{\partial x^2}; \quad \dfrac{\partial^3 u}{\partial x^2 \partial t} \in L^2(\Omega); \quad u(x,t)|_{x=0} = \dfrac{\partial u(x,t)}{\partial x}|_{x=0} = \int_\alpha^l u(x,t)dx = 0\}$.
•E_λ is a space of Banach which is the completeness of $D(\mathcal{L}_\lambda)$ in relation to the norm:

$$\|u\|_{E_\lambda^2} = \int_\Omega (l-x)\left[\frac{1}{2}|\frac{\partial u}{\partial t}|^2 + |\frac{\partial^2 u}{\partial t \partial x}|^2\right] dxdt + \sup_{0\le t\le T}\int_0^2 (l-x)|\frac{\partial u}{\partial x}|^2 dx. \quad (10)$$

•F is the Hilbert space with the norm.

$$\|\mathcal{L}_\lambda u\|_F^2 = \int_\Omega |f(x,t)|^2 dxdt + \int_0^l \left[|\varphi(x)|^2 + |\varphi'(x)|^2\right] dx. \quad (11)$$

Definition 2.1. *We call a strong generalized solution of the problem* (5) *to* (8) *with the conditions* 3 *and* 4*, a solution of equation:*

$$\overline{\mathcal{L}}u = F \quad (12)$$

and (6) *to* (8) *with the conditions* 3 *and* 4 *where* $\overline{\mathcal{L}}$ *is the closure of* $\mathcal{L}$.

Lemma 2.2. *Consider the operator* M *defined by:* $Mv = \psi_\alpha v - Jv$ *where*

$$\psi_\alpha(x) = \begin{cases} 1, \ si \quad 0 \le x \le \alpha \\ \dfrac{l-x}{l-\alpha}, \ si \quad \alpha \le x \le l \end{cases} \qquad J_v = \begin{cases} 0, \ si \quad 0 \le x \le \alpha \\ \dfrac{1}{l-\alpha}\int_x^l v(y)dy, \ si \alpha \le x \le l \end{cases}$$

and v *a positive function.*

We have:

1)

$$\forall x \in (0,l), 0 < \frac{l-x}{l-\alpha} \le \psi_\alpha(x) \le 1. \quad (14)$$

2)

$$\forall v \in D(\mathcal{L}), Mv \le \psi_\alpha v \quad et \quad Mv \le v. \quad (15)$$

3)
$\forall u$ and v elements of $D(\mathcal{L})$ and $A(x,t)$ defined on $(0,l)\times(0,T)$, we have a:

$$-\int_0^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx = \int_0^l \psi_\alpha(x)A(x,t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx. \tag{16}$$

Proof.
1) (14) comes from the definition of ψ_α.
2) J *et* ψ_α are positive by definition, hence $Mv \le \psi_\alpha v, \forall v \in D(\mathcal{L})$.
And because $0 < \psi_\alpha(x) \le 1$, it comes:$Mv \le v$.
3) $\forall u$ and v elements de $D(\mathcal{L})$ we have :
$-\int_0^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx = -\int_0^\alpha \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx - \int_\alpha^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx$
Let's first calculate $\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx$.

$$\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)(\Psi_\alpha(x)v - Jv(x))\,dx$$

$$\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)vdx.$$

By integrating, in parts, we have:

$$\int_0^\alpha \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = -\int_0^\alpha A(x,t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx + \left[\left(A(x,t)\frac{\partial u}{\partial x}\right)v\right]_0^\alpha.$$

Let us now calculate $\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx$.

$$\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)(\Psi_\alpha(x)v - Jv(x))\,dx$$

$$\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \frac{1}{l-\alpha}\left\{\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)(l-x)vdx - \right.$$
$$\left. -\int_\alpha^l \left\{\frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)\int_x^l v(y)dy\right\}dx\right\}.$$

By integrating, in parts, the two terms on the right, it comes

$$\int_\alpha^l \frac{\partial}{\partial x}\left(A(x,t)\frac{\partial u}{\partial x}\right)Mvdx = \frac{1}{l-\alpha}\left[\left[A(x,t)\frac{\partial u}{\partial x}(l-x)v\right]_0^l - \right.$$
$$\left. -\int_\alpha^l A(x,t)\frac{\partial u}{\partial x}\left(-v + (l-x)\frac{\partial v}{\partial x}\right)dx\right] -$$

$$-\frac{1}{l-\alpha}\left[\left[A(x,t)\frac{\partial u}{\partial x}\int_x^l v(y)dy\right]_\alpha^l + \int_\alpha^l A(x,t)\frac{\partial u}{\partial x}v(x)dx\right]$$

$$-\int_0^l \frac{\partial}{\partial x}(A(x,t)\frac{\partial u}{\partial x})Mvdx = \int_0^l \psi_\alpha(x)A(x,t)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx.$$

□

Lemma 2.3. *If* $g \in D(\mathcal{L})$ *and verifies* (5) - (7) *then:*

$$\int_0^l Jg.gdx = 0. \tag{18}$$

Proof.

$$\begin{aligned}
\int_0^l Jg.gdx &= \int_\alpha^l Jg.gdx \\
&= \frac{1}{l-\alpha}\int_\alpha^l \left[g(x)\int_x^l g(y)dy\right]dx \\
&= -\frac{1}{l-\alpha}\int_\alpha^l \left[\frac{d}{dx}\left(\int_x^l g(y)dy\right)\int_x^l g(y)dy\right]dx \\
&= -\frac{1}{l-\alpha}\int_\alpha^l \frac{1}{2}\left[\frac{d}{dx}\left(\int_x^l g(y)dy\right)\right]^2 dx \\
&= -\frac{1}{2(l-\alpha)}\left[\int_x^l g(y)dy\right]^2 \text{ then} \\
\int_0^l Jg.gdx &= 0.
\end{aligned}$$

□

2.2 Energetic inequality

Theorem 2.4. *If the conditions* 3 *and* 4 *are satisfied then there exists a positive constant* C *such that, for any function* $u \in D(\mathcal{L})$ *solution of the problem* (5) *to* (8)*, we have:*

$$\|u\|_{E_\lambda} \leq C\|\mathcal{L}_\lambda u\|_F. \tag{19}$$

Proof. Let $u \in D(\mathcal{L})$ function and check (5) to (8)) and the conditions 3 and 4.

By multiplying (5) by $M\dfrac{\partial u}{\partial t}$ and integrating with respect to x on $(0,l)$ and then with respect to t of 0 to τ, it comes:

$$\int_0^\tau\int_0^l \mathcal{L}_\lambda u M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \frac{\partial u}{\partial t}M\frac{\partial u}{\partial t}dxdt - \lambda\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dxdt -$$
$$-\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dxdt. \quad (E_1)$$

Let us express each term of the right-hand side of the equality (E_1)

• $\int_0^\tau\int_0^l \frac{\partial u}{\partial t}M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \frac{\partial u}{\partial t}\psi_\alpha(x)\frac{\partial u}{\partial t}dxdt - \int_0^\tau\int_0^l \frac{\partial u}{\partial t}J\frac{\partial u}{\partial t}dxdt.$
As $u\in D(\mathcal{L})$ then $\frac{\partial u}{\partial t}\in L^2(\Omega)$ and after (18), we have: $\int_0^l \frac{\partial u}{\partial t}J\frac{\partial u}{\partial t}dx = 0$

Therefore $\int_0^\tau\int_0^l \frac{\partial u}{\partial t}M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \psi_\alpha(x)|\frac{\partial u}{\partial t}|^2|dxdt \quad (E_2)$

• $-\int_0^l \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dx = \int_0^l \psi_\alpha(x)b(x,t)\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dx.$

By replacing (16) $A(x), u$ and v respectivily by $b(x,t), \dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial t}$ and by integrating with respect to t from 0 to τ, we obtain:

$$-\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 u}{\partial x\partial t}\right)M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \psi_\alpha(x)b(x,t)|\frac{\partial^2 u}{\partial x\partial t}|^2 dxdt. \qquad (E_3)$$

• Similarly, replacing in (16) $A(x)$ and v respectivily by $a(x,t)$ and $\dfrac{\partial u}{\partial t}$, and by integrating with respect to t from 0 to τ, it comes:

$$-\int_0^\tau\int_0^l \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)M\frac{\partial u}{\partial t}dxdt = \int_0^\tau\int_0^l \psi_\alpha(x)a(x,t)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dxdt.$$

But $\dfrac{\partial u}{\partial x}\dfrac{\partial^2 u}{\partial x\partial t} = \frac{1}{2}\dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial x}\right)^2$ so:

$$
\begin{aligned}
-\int_0^{\tau}\int_0^{l}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)M\frac{\partial u}{\partial t}dxdt &= \frac{1}{2}\int_0^{\tau}\int_0^{l}\psi_\alpha(x)a(x,t)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dxdt \\
&= \frac{1}{2}\int_0^{l}\psi_\alpha(x)\left\{\int_0^{\tau}a(x,t)\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dt\right\}dx \\
&= \frac{1}{2}\int_0^{l}\psi_\alpha(x)\left\{\left[a(x,t)\left(\frac{\partial u}{\partial x}\right)^2\right]_0^{\tau}\right. \\
&\quad \left.-\int_0^{\tau}\frac{\partial a(x,t)}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dt\right\}dx \\
&= \frac{1}{2}\int_0^{l}\psi_\alpha(x)\left\{a(x,t)|\frac{\partial u}{\partial x}(x,\tau)|^2\right. \\
&\quad \left.-a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2\right\}dx - \\
&\quad \frac{1}{2}\int_0^{\tau}\int_0^{l}\psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2 dxdt \\
&= \frac{1}{2}\int_0^{l}\psi_\alpha(x)a(x,\tau)|\frac{\partial u}{\partial x}(x,\tau)|^2 dx \\
&\quad -\frac{1}{2}\int_0^{l}\psi_\alpha(x)a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2 dx \\
&\quad \frac{1}{2}\int_0^{\tau}\int_0^{l}\psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2 dxdt. \qquad (E_4)
\end{aligned}
$$

By using (E_2),(E_3) and (E_4) the equality (E_1) becomes:

$\int_0^{\tau}\mathcal{L}_\lambda uM\frac{\partial u}{\partial t}dxdt = \int_0^{\tau}\int_0^{l}\psi_\alpha(x)|\frac{\partial u}{\partial t}|^2 dxdt + \lambda\int_0^{\tau}\int_0^{l}\psi_\alpha(x)b(x,t)|\frac{\partial^2 u}{\partial x\partial t}|^2 dxdt +$
$\frac{1}{2}\int_0^{l}\psi_\alpha(x)a(x,\tau)|\frac{\partial u}{\partial x}(x,\tau)|^2 dx - \frac{1}{2}\int_0^{l}\psi_\alpha(x)a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2 dx-$
$-\frac{1}{2}\int_0^{\tau}\int_0^{l}\psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2 dxdt.(E_5)$

According to the inequality of Cauchy Bougnyakosky,

$2\int_0^{\tau}\mathcal{L}_\lambda uM\frac{\partial u}{\partial t}dxdt \leq 2\sqrt{\int_0^{\tau}\int_0^{l}(\mathcal{L}_\lambda u)^2 dxdt}\sqrt{\int_0^{\tau}\int_0^{l}\left(M\frac{\partial u}{\partial t}\right)^2 dxdt}$

the application of the inequality $2ab \leq a^2 + b^2$ to the right-hand side of the above-mentioned one makes it possible to obtain:

$$2\sqrt{\int_0^{\tau}\int_0^{l}(\mathcal{L}_\lambda u)^2 dxdt}\sqrt{\int_0^{\tau}\int_0^{l}\left(M\frac{\partial u}{\partial t}\right)^2 dxdt} \leq \int_0^{\tau}\int_0^{l}(\mathcal{L}_\lambda u)^2 dxdt + \int_0^{\tau}\int_0^{l}\left(M\frac{\partial u}{\partial t}\right)^2 dxdt.$$

We can deduce:$\int_0^{\tau}\int_0^{l}\mathcal{L}_\lambda uM\frac{\partial u}{\partial t}dxdt \leq \frac{1}{2}\int_0^{\tau}\int_0^{l}(\mathcal{L}_\lambda u)^2 dxdt + \frac{1}{2}\int_0^{\tau}\int_0^{l}\left(M\frac{\partial u}{\partial t}\right)^2 dxdt.$

(E_6)

But $\int_0^\tau \int_0^l \left(M\frac{\partial u}{\partial t}\right)^2 dxdt \leq \int_0^\tau \int_0^l \frac{\partial u}{\partial t} M \frac{\partial u}{\partial t} dxdt \leq \int_0^\tau \int_0^l \psi_\alpha(x) \left(\frac{\partial u}{\partial t}\right)^2 dxdt. \quad (E_7)$

Applying (10) to $\frac{\partial u}{\partial t}$ we have:$M\frac{\partial u}{\partial t} \leq \frac{\partial u}{\partial t}$ et $M\frac{\partial u}{\partial t} \leq \psi_\alpha \frac{\partial u}{\partial t}$.Thus the inequality (E_6) becomes:

$$\int_0^\tau \int_0^l \mathcal{L}_\lambda u M \frac{\partial u}{\partial t} dxdt \leq \frac{1}{2}\int_0^\tau \int_0^l (\mathcal{L}_\lambda u)^2 \, dxdt + \frac{1}{2}\int_0^\tau \int_0^l \psi_\alpha(x) \left(\frac{\partial u}{\partial t}\right)^2 dxdt. \quad (E_8)$$

By replacing in the relation (E_8) the left term by its expression of the equality (E_5), one obtains:

$$\int_0^\tau \int_0^l \psi_\alpha(x)|\frac{\partial u}{\partial t}|^2 dxdt + \lambda \int_0^\tau \int_0^l \psi_\alpha(x) b(x,t)|\frac{\partial^2 u}{\partial x \partial t}|^2 dxdt + \frac{1}{2}\int_0^l \psi_\alpha(x) a(x,\tau)|\frac{\partial u}{\partial x}(x,\tau)|^2 dx -$$

$$\frac{1}{2}\int_0^l \psi_\alpha(x) a(x,0)|\frac{\partial u}{\partial x}(x,0)|^2 dx - \frac{1}{2}\int_0^\tau \int_0^l \psi_\alpha(x) \frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2 dxd \leq \frac{1}{2}\int_0^\tau \int_0^l (\mathcal{L}_\lambda u)^2 dxdt +$$

$$+\frac{1}{2}\int_0^\tau \int_0^l \psi_\alpha(x)(\frac{\partial u}{\partial t})^2 dxdt.$$

It then comes:

$$\int_0^\tau \int_0^l \psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda b(x,t)|\frac{\partial^2 u}{\partial x \partial t}|^2\right] dxdt + \frac{1}{2}\int_0^l \psi_\alpha(x) a(x,\tau)|\frac{\partial u)}{\partial x}(x,\tau)|^2 dx \leq$$

$$\leq \frac{1}{2}\int_0^\tau \int_0^l |\mathcal{L}_\lambda u|^2 dxdt + \frac{1}{2}\int_0^\tau \int_0^l \psi_\alpha(x)\frac{\partial a(x,t)}{\partial t}|\frac{\partial u}{\partial x}|^2 dxdt.(E_9)$$

However, according to (6), $u(x,0) = \phi(x)$ therefore $\frac{\partial u}{\partial x}(x,0) = \phi'(x)$. In addition, using **condition** 3, especially
$a_0 \leq a(x,t) \leq a_1$; $\frac{\partial a}{\partial t}(x,t) \leq a_2; 1 \leq b(x,t) \leq b_1$, (E_9) becomes:

$$\int_0^\tau \int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x \partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x \partial t}(x,\tau)\right|^2 dx \leqslant$$

$$\leqslant \frac{1}{2}\int_0^\tau \int_0^l |\mathcal{L}_\lambda u|^2 \, dxdt + \frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx + \frac{a_2}{2}\int_0^\tau \int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt. \quad (E_{10})$$

As

$\Psi_\alpha(x)$ et $\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x \partial t}\right|^2\right]$ are positive, by removing the first term from the left member of $E(10)$ we get:

$$\frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant \frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + \frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx+$$

$$+\frac{a_2}{2}\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt,$$

which can also be written in the form :

$$\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant \frac{1}{a_0}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx+$$

$$\frac{a_2}{a_0}\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt.(E_{11})$$

By setting

$$h(t) = \frac{1}{a_0}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx,$$

$c = \dfrac{a_2}{a_0}$ and $g(\tau) = \int_0^l \Psi_\alpha(x)\left|\dfrac{\partial u}{\partial x}(x,\tau)\right|^2 dx$, the inequality (E_{11}) can be written in the form :

$$g(\tau) \leqslant c\int_0^\tau g(t)dt + h(t)$$

and using Grönwall's inequality which states:Let g and h be two integrable functions such that $g(t) \geq 0$, $h(t) \geq 0$ and h increasing on $(0,T)$. If $g(\tau) \leq c\int_0^\tau g(t)dt + h(\tau)$ then: $g(\tau) \leq exp(c\tau)h(\tau)$ where $c \in \Re_+^*$,
of the inequality (E_{11}), we deduce :

$$\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant e^{\frac{a_2}{a_0}\tau}\left\{\frac{1}{a_0}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.$$

As $\tau \in [0;T]$, then $e^{\frac{a_2}{a_0}\tau} \leqslant e^{\frac{a_2}{a_0}T}$ and $\int_0^\tau .dt \leqslant \int_0^T$; then the last inequality can be written as

$$\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant e^{\frac{a_2}{a_0}T}\left\{\frac{1}{a_0}\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + \frac{a_1}{a_0}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}. \quad (E_{12})$$

By replacing in (E_{12}), τ by y then integrating with respect to y from 0 to τ we get

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,y)\right|^2 dxdy \leqslant e^{\frac{a_2}{a_0}T}\left\{\frac{1}{a_0}\int_0^\tau\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\, dxdtdy+\right.$$

$$\left.+\frac{a_1}{a_0}\int_0^\tau\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dxdy\right\}.(E_{13})$$

Replacing y by t in the left-hand member and applying $\int_0^\tau \leqslant \int_0^T$ to the member of (E_{13}) then multiplying on both sides by $\frac{a_2}{2}$ we find

$$\frac{a_2}{2}\int_0^\tau\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dxdt \leqslant \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + +a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.(E_{14})$$

When (E_{14}) is used to bound above the right hand side of (E_{10}), we get

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant$$

$$\frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\} + +\frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx.$$

And, again using inequality $\int_0^\tau .dt \leqslant \int_0^T .dt$, we have

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant$$

$$\leqslant \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\} + +\left\{\frac{1}{2}\int_0^\tau\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + \frac{a_1}{2}\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\};$$

which leads to

$$\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx \leqslant$$

$$\leqslant \left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.(E_{15})$$

Note that the right hand side does not depend on τ. In this way,

$$\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^\tau\int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx\right\} \leqslant$$

$$\leqslant \left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T\int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\},$$

which leads to

$$\int_0^\tau \int_0^l \Psi_\alpha(x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2}\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x\partial t}(x,\tau)\right|^2 dx\right\} \leqslant$$

$$\leqslant \left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T \int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\}.(E_{16})$$

From (1.9) the inequality gives

$$\frac{1}{l-\alpha}\int_0^\tau \int_0^l (l-x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \frac{a_0}{2(l-\alpha)}\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2 dx\right\} \leqslant$$

$$\leqslant \left(\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right)\left\{\int_0^T \int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + a_1\int_0^l |\varphi'(x)|^2 dx\right\}.(E_{17})$$

From this we deduce that

$$\min\left(\frac{1}{l-\alpha};\frac{a_0}{2(l-\alpha)}\right)\left\{\int_0^\tau \int_0^l (l-x)\left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + \lambda\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2\right] dxdt + \right.$$

$$\left.\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^l \Psi_\alpha(x)\left|\frac{\partial u}{\partial x}(x,\tau)\right|^2 dx\right\}\right\} \leqslant$$

$$\max(1;a_1)\left[\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right]\left\{\int_0^T \int_0^l |\mathcal{L}_\lambda u|^2\, dxdt + a_1\int_0^l \Psi_\alpha(x)|\varphi'(x)|^2 dx\right\},$$

hence

$$||u||_{E_\lambda} \leqslant C||\mathcal{L}_\lambda||_F, \text{ avec } C = \frac{\max(1;a_1)}{\min\left(\frac{1}{l-\alpha};\frac{a_0}{2(l-\alpha)}\right)}\left[\frac{1}{2} + \frac{a_2}{2a_0}Te^{\frac{a_2}{a_0}T}\right].$$

□

We know that for every function $g \in C[0,l]$ the following equality is true

$$g(l) = \lim_{\alpha\longrightarrow l}\frac{1}{l-\alpha}\int_\alpha^l g(x)dx.$$

Therefore, the problem (3), (4) is the limit when $\alpha \to l$ of all problems (1), (2).

In the present work we establish the a priori estimate for the difference $u_\alpha - u$ and using this estimate, we will prove that if $\alpha \longrightarrow l$ and $\varphi_\alpha \longrightarrow \varphi$, then $u_\alpha \longrightarrow u$.

3 The a priori estimate

These are the main results of the paper.

Theorem 3.1.
The conditions 1 *and* 2 *are satisfied.Then there exists the constant* C *independent of* u_α*,* u *and* α*, such that there exists the a priori estimate*

$$\int_0^l \int_0^T (l-x)\left[\left|\frac{\partial u_\alpha}{\partial t}-\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial^2 u_\alpha}{\partial t\partial x}-\frac{\partial^2 u}{\partial t\partial x}\right|^2\right]dxdt+\sup_{0\leqslant t\leqslant T}\int_0^l (l-x)\left|\frac{\partial u_\alpha}{\partial t}-\frac{\partial u}{\partial t}\right|^2 dx\leqslant$$

$$\leqslant C\left[\int_0^l \left|\varphi_\alpha'(x)-\varphi'(x)\right|^2 dx+\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,0)d\xi-h(0)\right|^2+\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right|^2 dt+\right.$$

$$\left.+\sup_{0\leqslant t\leqslant T}\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right|^2+\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l \frac{\partial u(\xi,t)}{\partial t}d\xi-h'(t)\right|^2 dt\right]. \quad (24)$$

Proof. Consider the problem (1), (2) and make the change of the unknown function:

$$u_\alpha(x,t)=v_\alpha(x,t)+\frac{2x}{l-\alpha}h(t) \quad (25)$$

Where $v_\alpha(x,t)$ are solutions to problems
$\frac{\partial v_\alpha}{\partial t}-\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 v_\alpha}{\partial x\partial t}\right)-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v_\alpha}{\partial x}\right)=f(x,t)+\frac{2}{l+\alpha}\frac{\partial b}{\partial x}h'(t)+\frac{2}{l+\alpha}\frac{\partial a}{\partial x}h(t)-\frac{2x}{l+\alpha}h'(t),$
$v_\alpha(x,0)=\varphi_\alpha(x)-\frac{2x}{l+\alpha}h(0),\ \frac{1}{l-\alpha}\int_\alpha^l v_\alpha(x,t)dx=0,\ v_\alpha(0,t)=0.$
In the problem (3) , (4) make the change of the unknown function

$$u=v+\frac{2x}{l^2-\alpha^2}\int_\alpha^l u(\xi,t)d\xi \quad (26)$$

where $v(x,t)$ is the solution of the problem
$\frac{\partial v}{\partial t}-\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 v}{\partial x\partial t}\right)-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}\right)=f(x,t)+\frac{2}{l^2+\alpha^2}\frac{\partial b}{\partial x}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi+$
$+\frac{2}{l^2-\alpha^2}\frac{\partial a}{\partial x}\int_\alpha^l u(\xi,t)d\xi-\frac{2x}{l^2-\alpha^2}\int_\alpha^l\frac{\partial u(\xi,t)}{\partial t}d\xi,$
$v(x,0)=\varphi(x)-\frac{2x}{l^2-\alpha^2}\int_\alpha^l u(\xi,0)d\xi$
$v(0,t)=0,\ \frac{1}{l-\alpha}\int_\alpha^l v(x,t)dx=0$
Then the function $w_\alpha=v-v_\alpha$ is solution of the problem

$$\frac{\partial w_\alpha}{\partial t}-\frac{\partial}{\partial x}\left(b(x,t)\frac{\partial^2 w_\alpha}{\partial x\partial t}\right)-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial w_\alpha}{\partial x}\right)=F_\alpha(x,t) \quad (27)$$

$$w_\alpha(x,0)=\phi_\alpha(x), \quad w_\alpha(0,t)=0, \quad \frac{1}{l-\alpha}\int_\alpha^l w_\alpha(x,t)dx=0 \tag{28}$$

where

$$F_\alpha(x,t)=\frac{\partial b}{\partial x}\frac{2}{l-\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l \frac{\partial u(\xi,t)}{\partial t}d\xi-h'(t)\right]+\frac{\partial a}{\partial x}\frac{2}{l+\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right]-$$

$$-\frac{2x}{l+\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l \frac{\partial u(\xi,t)}{\partial t}d\xi-h'(t)\right], \tag{30}$$

$$\phi_\alpha(x)=\varphi(x)-\varphi_\alpha-\frac{2x}{l+\alpha}\left(\frac{1}{l-\alpha}\int_\alpha^l u(\xi,0)d\xi-h(0)\right). \tag{31}$$

In preliminary for the problem (27), (28), it has been demonstrated that the following a priori estimate

$$\int_\Omega (l-x)\left[\left|\frac{\partial w_\alpha}{\partial t}\right|^2+\left|\frac{\partial^2 w_\alpha}{\partial x\partial t}\right|^2\right]dxdt+\sup_{0\leqslant t\leqslant T}\int_0^l (l-x)\left|\frac{\partial w_\alpha}{\partial x}\right|^2 dx\leqslant$$

$$\leqslant C_1\left\{\int_0^l\left|\frac{d\phi_\alpha}{dx}\right|^2 dx+\int_\Omega |F_\alpha(x,t)|^2 dxdt\right\}, \tag{33}$$

where the constant C_1 does not depend on $w_\alpha, \quad \phi_\alpha, \quad F_\alpha$.

From Equality

$$u-u_\alpha=w_\alpha+\frac{2x}{l+\alpha}\left[\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi-h(t)\right]$$

it results in inequalities

$$\int_\Omega (l-x)|\frac{\partial u}{\partial t}-\frac{\partial u_\alpha}{\partial t}|^2 dxdt\leqslant 2\int_\Omega (l-x)|\frac{\partial w_\alpha}{\partial t}|^2 dxdt+$$

$$+\frac{7l^2}{3}\int_0^T |h'(t)-\frac{1}{l-\alpha}\int_\alpha^l \frac{\partial u(\xi,t)}{\partial t}d\xi|^2 dt \tag{35}$$

$$\int_\Omega (l-x)\left|\frac{\partial^2 u}{\partial x\partial t}-\frac{\partial^2 u_\alpha}{\partial x\partial t}\right|^2 dxdt\leqslant 2\int_\Omega (l-x)\left|\frac{\partial^2 w_\alpha}{\partial x\partial t}\right|^2 dxdt+$$

$$+2\int_0^T\left|h'(t)-\frac{1}{l-\alpha}\int_\alpha^l \frac{\partial u(\xi,t}{\partial t}d\xi\right|^2 dt \tag{37}$$

$$\sup_{0\leqslant t\leqslant T}\int_0^l (l-x)\left|\frac{\partial u}{\partial x}-\frac{\partial u_\alpha}{\partial x}\right|^2 dx\leqslant 2\sup_{0\leqslant t\leqslant T}\int_0^l (l-x)\left|\frac{\partial w_\alpha}{\partial x}\right|^2 dx+$$

$$+4\sup_{0\leqslant t\leqslant T}\left|h(t)-\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi\right|^2. \tag{39}$$

of both ties (30),and (31),it results in inequalities

$$\int_{\Omega} |F_\alpha(x,t)|^2 \, dx \leqslant \frac{12}{l} a_2 \int_0^T \left| \frac{1}{l-\alpha} \int_\alpha^l u(\xi,t) d\xi - h(t) \right| dt +$$

$$12 \left(\frac{b_2}{l} + \frac{l}{3} \right) \int_0^T \left| \frac{1}{l-\alpha} \int_\alpha^l \frac{\partial u(\xi,t)}{\partial \xi} d\xi - h'(t) \right| dt, \tag{41}$$

$$\int_0^l \left| \frac{d\phi_\alpha}{dx} \right|^2 dx \leqslant 2 \int_0^l |\varphi'(x) - \varphi'_\alpha(x)|^2 \, dx + \frac{8}{l} \left| \frac{1}{l-\alpha} \int_\alpha^l u(\xi,0) d\xi - h(0) \right|^2 . \tag{42}$$

On the basis of the inequalities (35) - (42), the inequality (33) implies the inequality (24) in which
$C = \max \left(\frac{7}{3} l^2 + 2 + 6C_1 \left(\frac{4b_2}{l} + \frac{4l}{3} \right), 8 \frac{C_1 a_2}{l}, 4, \frac{16 C_1}{l} \right)$.
The theorem 3.1 is thus proved. □

4 Continuous dependence of solutions of mixed problems of the form of boundary conditions

Using the estimation (24) for the difference $u_\alpha - u$ of the solutions u_α of the problems (1), (2) with integral conditions and the solution u of the mixed problem (3), (4) with the local condition, we obtain the following result.

Theorem 4.1. *Let the conditions* 1 *and* 2.
If

$$\lim_{\alpha \longrightarrow l} \int_0^l |\varphi'_\alpha(x) - \varphi'(x)|^2 \, dx = 0 \tag{43}$$

then

$$\lim_{\alpha \longrightarrow 0} \left\{ \int_0^l \int_0^T (l-x) \left[\left| \frac{\partial u_\alpha}{\partial t} - \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial^2 u_\alpha}{\partial x \partial t} - \frac{\partial^2 u}{\partial x \partial t} \right|^2 \right] dx dt + \right.$$

$$\left. + \sup_{0 \leqslant t \leqslant T} \int_0^l (l-x) \left| \frac{\partial u_\alpha}{\partial x} - \frac{\partial u}{\partial x} \right|^2 dx \right\} = 0. \tag{45}$$

Proof. As $h(t) = u(l,t)$, then

$$\lim_{\alpha\longrightarrow l}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi - h(t)\right|^2 = \lim_{\alpha\longrightarrow l}\sup_{0\leqslant t\leqslant T}\left|\frac{1}{l-\alpha}\int_\alpha^l u(\xi,t)d\xi - u(l,t)\right|^2 = 0 \tag{46}$$

$$\lim_{\alpha\longrightarrow l}\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l \frac{\partial u(\xi,t)}{\partial t}d\xi - h'(t)\right|^2 dt = \lim_{\alpha\longrightarrow l}\int_0^T\left|\frac{1}{l-\alpha}\int_\alpha^l \frac{\partial u(\xi,t)}{\partial t} - \frac{\partial u(l,t)}{\partial t}\right|^2 dt = 0. \tag{47}$$

¿From the equality (24) and the equations (43), (46), (47), comes the equality (45).
This proves the theorem (3.1). To complete our search, let us show that for every $\varphi \in W_2^1(0,T)$ function satisfying the conditions $\varphi(0) = 0$ and $\varphi(l) = h(0)$,there exists $\varphi_\alpha \in W_2^1(0,T)$ the functions such that

$$\varphi_\alpha(0) = 0, \quad \frac{1}{l-\alpha}\int_\alpha^l \varphi_\alpha(x)dx = h(0) \tag{48}$$

and the equality (43) is checked.
Let
$\varphi_\alpha(x) = \varphi(x) - \frac{2x}{l+\alpha}\left(\frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - h(0)\right)$, then $\varphi_\alpha \in W_2^1(0,T), \varphi_\alpha(0) = 0$,

$$\frac{1}{l-\alpha}\int_\alpha^l \varphi_\alpha(x)dx = \frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - \frac{2}{l-\alpha}\int_\alpha^l \frac{2xdx}{l+\alpha}\left(\frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - h(0)\right) =$$
$$= \frac{1}{l-\alpha}\int_0^l \varphi(x)dx - \left(\frac{1}{l-\alpha}\int_\alpha^l \varphi(x)dx - h(0)\right) = h(0).$$

And

$$\lim_{\alpha\longrightarrow l}\int_0^l \left|\varphi'_\alpha - \varphi'(x)\right| dx = \lim_{\alpha\longrightarrow l}\frac{4l}{(l+\alpha)^2}\left|\int_0^l \frac{1}{l-\alpha}\varphi(x)dx - h(0)\right|^2 = \frac{4}{l}\left|\varphi(l) - h(0)\right|^2 = 0.$$

In other words, we have just demonstrated the continuous dependence of the solutions of the mixed problems for the pseudo-parabolic equations of the form of the boundary conditions. □

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