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The Equity Premium Puzzle Based on a Jump-Diffusion Model

Fanchao Zhou[1] and Yanyun Li, Jun Zhao, Peibiao Zhao[2]

Abstract

Gong and Zou [1] studied and explained the equity premium puzzle with the domestic output process satisfying a diffusion stochastic differential equation. In this paper, we further study and arrive at a positive solution of the equity premium puzzle based on the domestic output process satisfying a jump-diffusion stochastic differential equation. The conclusions obtained here can be regarded as a natural generalization of the work by Gong and Zou [1].

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Keywords: The equity premium puzzle; The spirit of capitalism; Jump; The stochastic growth model

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[1] Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China.
Supported by a Grant-in-Aid for Scientific Research from Nanjing University of Science and Technology(KN11008), and by NUST Research Funding No.30920140132035. E-mail:fczhou01forp@163.com

[2] Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China.
Supported by NNSF(11371194) and by a Grant-in-Aid for Science Research from Nanjing University of Science and Technology (2011YBXM120), by AD20370.
E-mail:1033587458@qq.com; pbzhao@njust.edu.cn

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1 Introduction

The equity premium puzzle was first put forward by Mr Mehra and Prescott in 1985, through an analysis of the American historical data over the past more than a century, they found that the return rate of stocks is 7.9%, and the return rate of the corresponding risk-free securities is only 1%, the premium is 6.9%. Furthermore, an analysis of the data for the other developed countries also indicated that there were different levels of premiums. Mehra and Prescott [2] called this phenomenon “an equity premium puzzle”.

The explanation for the equity premium puzzle can be simply summarized as two folds: the first fold is to explore the theoretical model, and find the inconsistencies with realities and modify them; the second fold is to find the causes and solutions to the equity premium puzzle from the empirical aspects.

Benartzi and Thaler [3], Barberis, Huang and Santos [4], et al, used the prospect theory to explain the equity premium puzzle; Camerer and Weber [5], Maenhout [6] explained the equity premium puzzle by the Ellsberg Paradox theory. In addition, Constantinides, Donaldson and Mehra [7] studied the equity premium puzzle based on the asset pricing with the borrowing constraints. McGrattan and Prescott [8] examined whether the general equilibrium economy implied the equity premium, and explained the equity premium based on the change of individual income tax rate. Rietz [9] introduced the small probability events which caused a decrease in consumption, and explained the equity premium. Heaton [10] and Lucas [10, 11] researched the equity premium based on the infinite-horizon model. Aiyagari and Getler [12] thought that the transaction cost gap between stock and bond markets lead to the equity premium. Brad Barber, and Odean [13] argued the equity premium puzzle by virtue of the return rate of stocks under the markets with various costs. In 2002, Gong and Zou based on the domestic output process satisfying a geometric brownian motion, studied the equity premium puzzle by the stochastic optimal control theory, and gave the asset-pricing relationships.

In this paper, similar to the work by Gong and Zou [1], we study the equity premium puzzle based on the domestic output process satisfying a jump-diffusion stochastic differential equation.

The paper is organized as follows. The first two sections briefly introduce some notations and terminologies. Section 3 proposes a model based on a stochastic jump-diffusion differential equation. Section 4 gives an example. Section 5 tries to explain the equity premium puzzle. The results posed here can be regarded as a natural generalization of Gong and Zou [1].

2 Preliminaries

Assume that there are two assets in the economy: the government bond, B, and the capital stock, K. Let the output Y (Gong and Zou [1], Eaton [14] and Turnovsky [15]) satisfy

$$dY = \alpha K dt + \alpha K dy,$$

where α is the marginal physical product of the capital stock K, and dy satisfies

$$E(dy) = 0, Var(dy) = \sigma_y^2 dt.$$

If the inflation rate is stochastic as in Fisher [16], then the return on the government bond B will also be subject to a stochastic process. In the period of time dt, it is assumed that the stochastic real rate of the return on the bond B, dR_B, is given by

$$dR_B = r_B dt + du_B,$$

where r_B and du_B will be determined endogenously by the macroeconomic equilibrium. The stochastic real rate of the return on the capital is

$$dR_K = \frac{dY}{K} = \alpha dt + \alpha dy \hat{=} r_K dt + du_K.$$

Without any loss of generality, the taxes are levied on the capital income and the consumption c, that is,

$$dT = (\tau r_K K + \tau_c c)dt + \tau' K du_K = (\tau \alpha K + \tau_c c)dt + \tau' \alpha K dy,$$

where τ, τ' are the tax rates on the deterministic component and the stochastic component of the capital income, respectively, and τ_c is the tax rate on the consumption.

Now, the representative agent's wealth W_t is the sum of the holdings of K_t and B_t,

$$W_t = K_t + B_t.$$

Let n_B and n_K represent the proportion of the wealth invested on the bond and the capital,

$$n_B = \frac{B_t}{W_t}, n_K = \frac{K_t}{W_t}, n_K + n_B = 1.$$

Now, the representative agent chooses the consumption-wealth ratio, $\frac{c}{W}$, and the portfolio shares, n_B and n_K, to maximize his expected utility subject to the budget constraint, i.e.,

$$\max E \int_0^\infty u(c, W_t) e^{-\beta t} dt$$

$$\begin{cases} \frac{dW_t}{W_t} = (n_B r_B + n_K(1-\tau) r_K - (1+\tau_c)c)dt + dw, \\ n_B + n_K = 1. \end{cases}$$

where β is the time discount rate, $dw = n_B du_B + n_K(1-\tau')du_K$.

3 The Stochastic Optimal Control Problem Based on a Jump-Diffusion Model

3.1 Basic Assumptions

Referring to the basic hypothesis of domestic output by Eaton [14] and Turnovsky [15], we renew the equation for the domestic output Y as follows

$$dY = \alpha K dt + \alpha K dy + \alpha K \varphi(t) dN(t), \tag{3.1}$$

where $N(t)$ is a poisson process defined on a probability space (Ω, F, P), $\varphi(t)$ is the jumping amplitude.

Assume that there are two assets in the economy: the government bond, B, and the capital stock, K with the following equation

$$dR_B = r_B dt + du_B, \tag{3.2}$$

where r_B and du_B are defined just as above. Then, the stochastic real rate of the return on the capital is

$$\begin{aligned} dR_K = \frac{dY}{K} &= \alpha dt + \alpha dy + \alpha\varphi(t)dN(t) \\ &= r_K dt + du_K + \alpha\varphi(t)dN(t). \end{aligned} \tag{3.3}$$

Without any loss of generality, the taxes are levied on the capital income and the consumption, that is,

$$\begin{aligned} dT &= (\tau r_K K + \tau_c c)dt + \tau' K du_K + \tau'' K\varphi(t)dN(t) \\ &= (\tau\alpha K + \tau_c c)dt + \tau'\alpha K dy + \tau'' K\varphi(t)dN(t), \end{aligned} \tag{3.4}$$

where τ, τ', τ'' are the tax rates on the deterministic part of the capital income, the normal component of the stochastic capital income and the abnormal component of the stochastic capital income, τ_c is the tax rate on the consumption c.

3.2 Wealth Process

For a period of time dt, there are the certain tax and the consumption, so the wealth in the moment t follows the stochastic differential equation

$$dW_t = n_B W_t dR_B + n_K W_t dR_K - dT - cdt. \tag{3.5}$$

Substituting (3.2), (3.3) and (3.4) into (3.5), we have

$$\begin{aligned} dW_t &= n_B W_t dR_B + n_K W_t dR_K - dT - cdt, \\ &= n_B W_t(r_B dt + du_B) + n_K W_t(r_K dt + du_K + \alpha\varphi(t)dN(t)) \\ &- [(\tau\alpha K + \tau_c c)dt + \tau'\alpha K dy + \tau'' K\varphi(t)dN(t)] - cdt. \end{aligned}$$

Simplifying Equation (3.5), we obtain

$$\begin{aligned} dW_t &= (n_B W_t r_B + n_K W_t(1-\tau)r_K - (1+\tau_c)c)dt \\ &+ W_t dw + n_K W_t(1-\tau'')r_K\varphi(t)dN(t). \end{aligned} \tag{3.6}$$

where $n_B + n_K = 1, dw = n_B du_B + n_K(1-\tau')du_K$.

3.3 The Stochastic Optimal Control Problem

Consider the optimization problem

$$\max E\int_0^\infty u(c, W_t)e^{-\beta t}dt$$

$$\begin{cases} \frac{dW_t}{W_t} = (\rho - (1+\tau_c)\frac{c}{W_t})dt + dw + n_K r_K(1-\tau'')\varphi(t)dN(t), \\ n_B + n_K = 1. \end{cases}$$

where $\rho = n_B r_B + n_K(1-\tau)r_K, dw = n_B du_B + n_K(1-\tau')du_K$, and $\sigma_w^2 = n_B^2\sigma_B^2 + n_K^2(1-\tau')^2\sigma_K^2 + 2n_B n_K(1-\tau')\sigma_{BK}$.

To solve the agent's optimization problem above, we define the value function

$$V(W_t,t) = \max E_t\int_0^\infty u(c,W_s)e^{-\beta s}ds \hat{=} e^{-\beta t}X(W).$$

According to the dynamic programming principle(Yong [17]) and a direct computation, we arrive at the HJB equation of the above problem as follows

$$\max_{c,n_B,n_K}\{u(c,W_t) - \beta X(W) + (\rho - (1+\tau_c)\frac{c}{W})WX_W + \frac{1}{2}\sigma_w^2 W^2 X_{WW} +$$
$$\lambda n_K r_K(1-\tau'')\varphi(t)WX_W + \frac{1}{2}\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)W^2X_{WW}\} = 0.$$

The corresponding Lagrangian function is

$$L(c,n_B,n_K,\eta)\hat{=}u(c,W_t) - \beta X(W) + (\rho - (1+\tau_c)\frac{c}{W})WX_W + \frac{1}{2}\sigma_w^2 W^2 X_{WW} +$$
$$\lambda n_K r_K(1-\tau'')\varphi(t)WX_W + \frac{1}{2}\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)W^2X_{WW} + \eta(1-n_K-n_B). \tag{3.7}$$

3.4 A Macroeconomic Equilibrium

Considering the derivatives of (3.7) for $\frac{c}{W}, n_B, n_K, \eta$, the optimal conditions can be obtained as follows

Proposition 3.1. *The first-order conditions for the optimization problem can be written as follows*

$$\frac{\partial u(c,W)}{\partial c} = (1+\tau_c)X_W, \tag{3.8}$$

$$(r_B W X_W - \eta)dt + cov(dw, du_B)W^2X_{WW} = 0, \tag{3.9}$$

$$\begin{aligned}&(r_K(1-\tau)WX_W - \eta)dt + cov(dw,(1-\tau')du_K)W^2X_{WW} +\\ &[\lambda r_K(1-\tau'')\varphi(t)WX_W + \lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)W^2X_{WW}]dt = 0,\end{aligned} \tag{3.10}$$

$$n_B + n_K = 1,$$

where η is the Lagrangian multiplier associated with the portfolio selection constraint $n_B + n_K = 1$, furthermore, the optimal solutions of the problem must satisfy the Bellman equation

$$u(c,W_t) - \beta X(W) + (\rho - (1+\tau_c)\frac{c}{W})WX_W + \frac{1}{2}\sigma_w^2 W^2 X_{WW} +$$
$$\lambda n_K r_K(1-\tau'')\varphi(t)WX_W + \frac{1}{2}\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)W^2X_{WW} = 0.$$

In order to obtain the equilibrium solution of the whole economic system, we discuss the government's actions as Turnovsky [15]. Apart from discussing the government's tax policy, we give the government expenditure as follows

$$dG = g\alpha K dt + \alpha K dz, \tag{3.11}$$

where g is the percentage of government expenditure accounting for output, dz is temporally independent, normally distributed, and

$$E(dz) = 0, Var(dz) = \sigma_z^2 dt.$$

The government budget constraints can be described as:

$$dB = BdR_B + dG - dT. \tag{3.12}$$

Substituting (3.2), (3.4) and (3.11) into (3.12), we have

$$dB = (r_B B + \alpha(g-\tau)K - \tau_c c)dt + Bdu_B + \alpha K dz - \tau' K\alpha dy - \alpha K\tau''\varphi(t)dN(t).$$

A balanced product market requires

$$dK = dY - cdt - dG.$$

Substituting (3.1), (3.11) into the formula above, we know

$$dK = (\alpha K - \alpha g K - c)dt + \alpha K(dy - dz) + \alpha K\varphi(t)dN(t).$$

Then we will get

$$\frac{dK}{K} = [\alpha(1-g) - \frac{c}{n_K W}]dt + \alpha(dy - dz) + \alpha\varphi(t)dN(t). \tag{3.13}$$

Proposition 3.2. *The equilibrium system of the economy can be summarized as*

$$\frac{dK}{K} = [\alpha(1-g) - \frac{c}{n_K W}]dt + \alpha(dy - dz) + \alpha\varphi(t)dN(t),$$

$$\frac{\partial u(c,W)}{\partial c} = (1+\tau_c)X_W,$$

$$(r_B W X_W - \eta)dt + cov(dw, du_B)W^2 X_{WW} = 0,$$

$$[r_K(1-\tau)WX_W - \eta + \lambda r_K(1-\tau'')\varphi(t)WX_W + \lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)W^2X_{WW}]$$
$$dt + cov(dw, (1-\tau')du_K)W^2X_{WW} = 0,$$

$$n_B + n_K = 1.$$

Proposition 3.3. *The normal fluctuation component of the stochastic return of bonds, du_B, and the total wealth, dw, are decided by the following formulas*

$$du_B = \frac{\alpha}{n_B}[(1 - n_K(1-\tau'))dy - dz], \tag{3.14}$$

$$dw = \alpha(dy - dz). \tag{3.15}$$

Proof. According to the inter-temporal invariance of portfolio shares(Benaviea [18]), we have

$$\frac{dW}{W} = \frac{dK}{K} = \frac{dB}{B}. \tag{3.16}$$

That is, the growth of all real assets is the same as the stochastic rate. Combining with (3.6), (3.13), (3.14) and (3.16), we get

$$\begin{aligned} dw &= n_B du_B + n_K(1-\tau')\alpha dy = \alpha(dy - dz) \\ &= \frac{1}{n_B}[n_B du_B + \alpha n_K(dz - \tau' dy)]. \end{aligned}$$

From the equations above, and $n_B + n_K = 1$, we will get

$$\begin{aligned} du_B &= \frac{\alpha}{n_B}[n_B(dy - dz) - n_K(dz - \tau' dy)] \\ &= \frac{\alpha}{n_B}[(1-n_K)dy - (1-n_K)dz - n_K dz + n_K \tau' dy] \\ &= \frac{\alpha}{n_B}[(1 - n_K(1-\tau'))dy - dz]. \end{aligned}$$

This ends the Proof of Proposition 3.3. □

4 An Explicit Example

In order to find the explicit solution of the optimal control problem as above, we will specify the utility function as in Bakshi and Chen [19] as follows

$$u(c, W) = \frac{c^{1-\gamma}}{1-\gamma} W^{-\theta}, \tag{4.1}$$

where $\frac{1}{\gamma} > 0$ is the elasticity of intertemporal substitution, furthermore, when $\gamma > 1, \theta \geq 0$, and when $0 < \gamma < 1, \theta < 0$.

It is obvious that there holds

$$|\frac{du/u}{dW/W}| = |\frac{du}{dW}\frac{W}{u}| = |\frac{\theta c^{1-\gamma}}{1-\gamma}W^{-\theta-1}\frac{W}{u}| = |\theta|.$$

In the existing theory, the wealth is no more valuable than the rewards of its implied consumption. In reality, the investors acquire the wealth not just for its implied consumption, but for the resulting social status. Max M. Weber [20] describes this desire for wealth as the spirit of capitalism.

$|\theta|$ measures the investor's concern with his social status or his spirit of capitalism. The larger the parameter, $|\theta|$, the stronger the agent's spirit of capitalism or concern with his social status.

Under the specific utility function (4.1), we can get

Proposition 4.1. *The first-order optimal conditions are*

$$\begin{aligned}\frac{c}{W} = &\frac{\beta + \frac{1}{2}\sigma_w^2(1-\gamma-\theta)(\gamma+\theta) - \rho(1-\gamma-\theta)}{\gamma(1+\tau_c)^{\frac{1-\gamma-\theta}{1-\gamma}}} \\ &- \lambda n_K r_K \varphi(t)(1-\tau'')\frac{1+\frac{1}{2}n_K r_K(1-\tau'')\varphi(t)(\gamma+\theta)}{\frac{\gamma(1+\tau_c)}{1-\gamma}},\end{aligned} \tag{4.2}$$

$$(r_B - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}})dt = (\gamma+\theta)cov(dw, du_B), \tag{4.3}$$

$$\begin{aligned}&(r_K(1-\tau) - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}})dt + (\lambda r_K(1-\tau'')\varphi(t) - \\ &\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta))dt = (\gamma+\theta)(1-\tau')cov(dw, du_K),\end{aligned} \tag{4.4}$$

where η is the Lagrangian multiplier, and

$$\begin{aligned}&\rho = n_B r_B + n_K(1-\tau)r_K, \\ &dw = n_B du_B + n_K(1-\tau')du_K, \\ &\sigma_w^2 = n_B^2\sigma_B^2 + n_B^2(1-\tau')^2\sigma_K^2 + 2n_B n_K(1-\tau')\sigma_{BK}.\end{aligned}$$

Proof. It is assumed that the form of the value function is

$$X(W) = \delta W^{1-\gamma-\theta}, \tag{4.5}$$

where δ is to be determined. Differentiating (4.5) with respect to W yields

$$\begin{aligned}&X_W = \delta(1-\gamma-\theta)W^{-\gamma-\theta}, \\ &X_{WW} = \delta(1-\gamma-\theta)(-\gamma-\theta)W^{-\gamma-\theta-1}.\end{aligned}$$

Now the first-order optimal conditions are

$$\frac{c}{W} = ((1+\tau_c)\delta(1-\gamma-\theta))^{-\frac{1}{\gamma}}, \tag{4.6}$$

$$\begin{aligned}&[r_B\delta(1-\gamma-\theta)W^{1-\gamma-\theta}-\eta]dt\\&+cov(dw,du_B)\delta(1-\gamma-\theta)(-\gamma-\theta)W^{1-\gamma-\theta}=0,\end{aligned} \tag{4.7}$$

$$\begin{aligned}&(r_K(1-\tau)\delta(1-\gamma-\theta)W^{1-\gamma-\theta}-\eta)dt+(\lambda r_K(1-\tau'')\varphi(t)\delta(1-\gamma-\theta)\\&W^{1-\gamma-\theta}-cov(dw,(1-\tau')du_K)\delta(1-\gamma-\theta)(\gamma+\theta)W^{1-\gamma-\theta}\\&-\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t))\delta(1-\gamma-\theta)(\gamma+\theta)W^{1-\gamma-\theta})dt=0.\end{aligned} \tag{4.8}$$

Replacing c in the Bellman equation with $W((1+\tau_c)\delta(1-\gamma-\theta))^{-\frac{1}{\gamma}}$, we have

$$\begin{aligned}&(\rho-(1+\tau_c)\frac{W((1+\tau_c)\delta(1-\gamma-\theta))^{-\frac{1}{\gamma}}}{W})W\delta(1-\gamma-\theta)W^{-\gamma-\theta}\\&-\beta\delta W^{1-\gamma-\theta}+\frac{1}{2}\sigma_w^2W^2\delta(1-\gamma-\theta)(-\gamma-\theta)W^{-\gamma-\theta-1}\\&+\frac{(W((1+\tau_c)\delta(1-\gamma-\theta))^{-\frac{1}{\gamma}})^{1-\gamma}}{1-\gamma}W^{-\theta}=0.\end{aligned}$$

Therefore,

$$\begin{aligned}((1+\tau_c)\delta(1-\gamma-\theta))^{-\frac{1}{\gamma}}&=\frac{\beta+\frac{1}{2}\sigma_w^2(1-\gamma-\theta)(\gamma+\theta)-\rho(1-\gamma-\theta)}{\frac{\gamma(1+\tau_c)(1-\gamma-\theta)}{1-\gamma}}\\&-\frac{\lambda n_K r_K(1-\tau'')\varphi(t)+\frac{1}{2}\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(1-\gamma-\theta)(\gamma+\theta)}{\frac{\gamma(1+\tau_c)}{1-\gamma}}.\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.6), one gets

$$\begin{aligned}\frac{c}{W}&=W((1+\tau_c)\delta(1-\gamma-\theta))^{-\frac{1}{\gamma}}\\&=\frac{\beta+\frac{1}{2}\sigma_w^2(1-\gamma-\theta)(\gamma+\theta)-\rho(1-\gamma-\theta)}{\frac{\gamma(1+\tau_c)(1-\gamma-\theta)}{1-\gamma}}\\&-\frac{\lambda n_K r_K(1-\tau'')\varphi(t)+\frac{1}{2}\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(1-\gamma-\theta)(\gamma+\theta)}{\frac{\gamma(1+\tau_c)}{1-\gamma}}.\end{aligned}$$

Dividing both sides of the equations (4.7) and (4.8) by $\delta(1-\gamma-\theta)W^{1-\gamma-\theta}$, we have

$$(r_B-\frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}})dt=(\gamma+\theta)cov(dw,du_B),$$

$$(r_K(1-\tau) - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}})dt + (\lambda r_K(1-\tau'')\varphi(t) -$$
$$\lambda n_K r_K^2 (1-\tau'')^2 \varphi^2(t)(\gamma+\theta))dt = (\gamma+\theta)(1-\tau')cov(dw, du_K).$$

This completes the Proof of Proposition 4.1. □

The formulas (4.3), (4.4) show the asset pricing relationships. $\frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}}$ can be regarded as the 'risk-free' return. (4.3) means that the return on bonds is equal to the 'risk-free' return plus a risk premium, which is proportional to the covariance between the total wealth and the bonds. Similarly, (4.4) means that the return on the capital is equal to the 'risk-free' return plus a risk premium, which is proportional to the covariance between the total wealth and the risky capital.

From Proposition 3.3, and the optimal conditions (4.3), (4.4) and (3.12), we have

$$\begin{aligned}
&\sigma_w^2 = \alpha^2(\sigma_y^2 + \sigma_z^2)dt, \\
&cov(dw, du_B) = \frac{\alpha^2}{n_B}[(1 - n_K(1-\tau'))\sigma_y^2 + \sigma_z^2]dt, \\
&cov(dw, (1-\tau')du_K) = \alpha^2(1-\tau')\sigma_y^2 dt.
\end{aligned}$$

Proposition 4.2. *The mean return on bonds and the stochastic growth rate of the economy are*

$$\begin{aligned}
r_B = \alpha(1-\tau) &+ \frac{\gamma+\theta}{n_B}\alpha^2(\tau'\sigma_y^2 + \sigma_z^2) + \lambda r_K(1-\tau'')\varphi(t) \\
&- \lambda n_K^2 r_K^2 (1-\tau'')^2 \varphi^2(t)(\gamma+\theta)
\end{aligned} \tag{4.10}$$

$$\phi = \frac{r_B n_B + (g-\tau)\alpha n_K + \tau_c \frac{c}{W}}{n_B} = \rho - (1+\tau_c)\frac{c}{W} \tag{4.11}$$

Proof. Noticing (4.4), we have

$$\begin{aligned}
\frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} &= \alpha(1-\tau) - (\gamma+\theta)(1-\tau')\alpha^2\sigma_y^2 \\
&\quad + \lambda r_K(1-\tau'')\varphi(t) - \lambda n_K r_K(1-\tau'')^2\varphi^2(t)(\gamma+\theta).
\end{aligned}$$

Substituting it into (4.3), we can get the formula (4.10). Equation (4.11) is obvious. □

5 The Interpretation of the Equity Premium Puzzle

In this section, we will discuss how to explain the equity premium puzzle by the existence of the spirit of capitalism. For simplicity, we set the consumption tax $\tau_c = 0$.

First, we give the equilibrium asset-pricing relationships. We define the market portfolio as $Q = n_B W + n_K W$, the return rate on the market portfolio is

$$\begin{aligned} r_Q = \rho = r_B n_B + r_K(1-\tau)n_K &= \alpha(1-\tau) + (\gamma+\theta)\alpha^2(\tau'\sigma_y^2 + \sigma_z^2) \\ &+ \lambda n_B r_K(1-\tau'')\varphi(t) - \lambda n_B n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta) \end{aligned}$$

Proposition 5.1. *The equilibrium asset-pricing relationships are*

$$r_B - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} = \beta_B\left(r_Q - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}}\right), \tag{5.1}$$

$$(1-\tau)r_K - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} = \beta_K\left(r_Q - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}}\right). \tag{5.2}$$

where

$$\beta_B = \frac{(\gamma+\theta)\frac{\alpha^2}{n_B}[(1-n_K(1-\tau'))\sigma_y^2 + \sigma_z^2]}{-\lambda r_K n_K(1-\tau'')\varphi(t) + \lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta) + (\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)},$$

$$\beta_K = \frac{-\lambda r_K(1-\tau'')\varphi(t) + \lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta) + (\gamma+\theta)(1-\tau')\alpha^2\sigma_y^2}{-\lambda r_K n_K(1-\tau'')\varphi(t) + \lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta) + (\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}.$$

Proof. Since

$$\begin{aligned} \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} &= \alpha(1-\tau) + \lambda r_K(1-\tau'')\varphi(t) \\ &- (\gamma+\theta)(1-\tau')\alpha^2\sigma_y^2 - \lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta), \end{aligned}$$

$$\begin{aligned} r_Q &= \alpha(1-\tau) + (\gamma+\theta)\alpha^2(\tau'\sigma_y^2 + \sigma_z^2) \\ &+ \lambda n_B r_K(1-\tau'')\varphi(t) - \lambda n_B n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta). \end{aligned}$$

we get

$$\begin{aligned} r_Q - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} &= \lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta) \\ &+ (\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2) - \lambda r_K n_K(1-\tau'')\varphi(t). \end{aligned}$$

By Proposition 4.1, we can come to the conclusion. □

Also $\frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}}$ can be regarded as the risk-free return. The formulas (5.1) and (5.2) imply that the risk-asset returns (the government bonds and capitals) are given by the familiar consumption-based capital asset pricing model with r_Q as the return on the market portfolio.

In addition, we define the return rate on the market portfolio in the absence of the spirit of capitalism as $\bar{r}_Q$. In our definition of r_Q, we can set $\theta = 0$. Therefore,

$$\bar{r}_Q = \alpha(1-\tau)+\lambda n_B r_K(1-\tau'')\varphi(t)-\lambda n_B n_K r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\tau'\sigma_y^2+\sigma_z^2).$$

At the same time, we have

$$\frac{\eta}{\delta(1-\gamma)W^{1-\gamma}} = \alpha(1-\tau)+\lambda r_K(1-\tau'')\varphi(t)-\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\gamma-\gamma(1-\tau')\alpha^2\sigma_y^2.$$

So we get the corresponding asset-pricing relationships as

$$\bar{r}_B - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}} = \bar{\beta}_B(\bar{r}_Q - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}),$$
$$(1-\tau)\bar{r}_K - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}} = \bar{\beta}_K(\bar{r}_Q - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}),$$

where $\bar{r}_B$ and $\bar{r}_K$ are the corresponding forms to r_B and r_K in the case $\theta = 0$, and

$$\bar{\beta}_B = \frac{\gamma\frac{\alpha^2}{n_B}[(1-n_K(1-\tau'))\sigma_y^2+\sigma_z^2]}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)},$$
$$\bar{\beta}_K = \frac{-\lambda r_K(1-\tau'')\varphi(t)+\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma(1-\tau')\alpha^2\sigma_y^2}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)}.$$

On the basis of Proposition 5.1, we can get the following corollary.

Corollary 5.2.

$$r_B - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} > \bar{r}_B - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}},$$
$$(1-\tau)r_K - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} > (1-\tau)\bar{r}_K - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}.$$

Proof. Here we only prove the case of $\theta > 0$. since

$$\frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}$$
$$= -\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t) - \theta(1-\tau')\alpha^2\sigma_y^2 < 0,$$

i.e,

$$\frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} < \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}.$$

and $\bar{r}_Q < r_Q$, we just need to prove $\bar{\beta}_B < \beta_B, \bar{\beta}_K < \beta_K$. In fact,

$$\begin{aligned}
\frac{\bar{\beta}_B}{\beta_B} &= \frac{\frac{\gamma \frac{\alpha^2}{n_B}[(1-n_K(1-\tau'))\sigma_y^2+\sigma_z^2]}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)}}{\frac{(\gamma+\theta)\frac{\alpha^2}{n_B}[(1-n_K(1-\tau'))\sigma_y^2+\sigma_z^2]}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}} \\
&= \frac{\gamma}{\gamma+\theta}\frac{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)} \\
&< \frac{\gamma}{\gamma+\theta}\frac{\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}{\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)} \\
&= \frac{\gamma}{\gamma+\theta}\frac{\gamma+\theta}{\gamma} = 1.
\end{aligned}$$

$\bar{\beta}_B < \beta_B$ is proved.

$$\begin{aligned}
\frac{\bar{\beta}_K}{\beta_K} &= \frac{\frac{-\lambda r_K(1-\tau'')\varphi(t)+\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma(1-\tau')\alpha^2\sigma_y^2}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)}}{\frac{-\lambda r_K(1-\tau'')\varphi(t)+\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)(1-\tau')\alpha^2\sigma_y^2}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}} \\
&= \frac{-\lambda r_K(1-\tau'')\varphi(t)+\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma(1-\tau')\alpha^2\sigma_y^2}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)} \cdot \\
&\quad \frac{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}{-\lambda r_K(1-\tau'')\varphi(t)+\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)(1-\tau')\alpha^2\sigma_y^2} \\
&= \frac{-\lambda r_K(1-\tau'')\varphi(t)+\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma(1-\tau')\alpha^2\sigma_y^2}{-\lambda r_K(1-\tau'')\varphi(t)+\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)(1-\tau')\alpha^2\sigma_y^2} \cdot \\
&\quad \frac{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}{-\lambda r_K n_K(1-\tau'')\varphi(t)+\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)} \\
&< \frac{\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma(1-\tau')\alpha^2\sigma_y^2}{\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)(1-\tau')\alpha^2\sigma_y^2} \cdot \\
&\quad \frac{\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)(\gamma+\theta)+(\gamma+\theta)\alpha^2(\sigma_y^2+\sigma_z^2)}{\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)\gamma+\gamma\alpha^2(\sigma_y^2+\sigma_z^2)} \\
&= \frac{\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)+(1-\tau')\alpha^2\sigma_y^2}{\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)+(1-\tau')\alpha^2\sigma_y^2} \cdot \frac{\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)+\alpha^2(\sigma_y^2+\sigma_z^2)}{\lambda n_K^2 r_K^2(1-\tau'')^2\varphi^2(t)+\alpha^2(\sigma_y^2+\sigma_z^2)} \\
&= 1,
\end{aligned}$$

$\bar{\beta}_K < \beta_K$ is proved. □

Remark 5.3. *If $\theta < 0$, the conclusion is opposite. That is*

$$r_B - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} < \bar{r}_B - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}},$$
$$(1-\tau)r_K - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} < (1-\tau)\bar{r}_K - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}.$$

In this situation, the existence of the spirit of capitalism plays a negative role, that is it makes the gap between the risky assets and risk-free assets narrowed. Referring to the concept of the equity premium puzzle in Bakshi and Chen [19], we can call this phenomenon equity underpricing puzzle.

Next, we make a comparison between the conclusion of this paper and the conclusion before, thus compare the premium magnitude of two models.

$$\begin{aligned} & r_B - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} - (\bar{r}_B - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}) \\ & = \theta\frac{\alpha^2}{n_B^2}[(1-n_K(1-\tau'))\sigma_y^2 + \sigma_z^2](1-\tau)r_K - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} \\ & - ((1-\tau)\bar{r}_K - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}) \\ & = \lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\theta + \theta(1-\tau')\alpha^2\sigma_y^2. \end{aligned} \tag{5.3}$$

The previous conclusion is

$$\begin{aligned} & r_B - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} - (\bar{r}_B - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}) \\ & = \theta\frac{\alpha^2}{n_B^2}[(1-n_K(1-\tau'))\sigma_y^2 + \sigma_z^2], \end{aligned} \tag{5.4}$$

$$\begin{aligned} & (1-\tau)r_K - \frac{\eta}{\delta(1-\gamma-\theta)W^{1-\gamma-\theta}} - ((1-\tau)\bar{r}_K - \frac{\eta}{\delta(1-\gamma)W^{1-\gamma}}) \\ & = \theta(1-\tau')\alpha^2\sigma_y^2. \end{aligned} \tag{5.5}$$

Due to the existence of difficulties in comparing the premium of government bond between the two models, so it isn't discussed in this paper. Consider (5.3)-(5.5), we have the following remark.

Remark 5.4. *The difference between the premium of capital stock in this paper and the original is $\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\theta$. If $\theta > 0$, $\lambda n_K r_K^2(1-\tau'')^2\varphi^2(t)\theta > 0$. This implies the result of this paper is obviously better than the original.*

The results posed here imply that, because of the existence of the spirit of capitalism, the gap of the return rate on risky assets and risk-free assets will *be enlarged. Like Bakshi and Chen [19], our findings can partially explain the equity premium puzzle in Mehra and Prescott [2] .*

6 Declaration

Competing interests
The authors declare that they have no competing interests.
Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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