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Powers of the generalized 2-Fibonacci matrices

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Abstract

In this paper, we investigate the closed formulas for the entries of the power of the 2×2 matrix obtained by generalized 2-step Fibonacci sequence.

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1 Introduction

Fibonacci numbers are one of the best-known numerical sequences and have many important applications to a wide variety of research areas such as mathematics, computer science, physics, biology, and statistics. For the applications and the theory of Fibonacci numbers see, e.g. [3, 6, 8, 9, 10, 12, 13, 16, 17] and the references given therein. In [3, 12], the well-known Fibonacci sequence is formulated by the recurrence relation $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, with $f_1 = f_2 = 1$.

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Many authors have considered and discussed the generalizing of the above definition as in the following:

- the k-step Fibonacci sequence is derived by the recurrence relation, $f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-k}$, $n \geq k+1$, with $f_1 = f_2 = \ldots = f_k = 1$, [1, 3, 10, 12],
- the generalized k-step Fibonacci sequence is derived by the recurrence relation, $f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k}$, $n \geq k+1$, with $f_1 = f_2 = \ldots = f_k = 1$, and $c_1, c_2, \ldots, c_k$ are arbitrary real numbers, [2, 6, 8, 9, 10, 11, 18].

Furthermore, important relations between the $k-$step Fibonacci numbers and the special matrices have been investigated; the determinants of the matrices constructed by $k-$step Fibonacci numbers are obtained in [10] and the properties of the determinants are discussed in [9], the sums of the generalized Fibonacci numbers are derived directly using the matrix representation and method in [2, 4, 5, 11]; some closed formulas for the generalized Fibonacci sequence are derived by matrix methods [8, 11, 13]. Recently, two limiting properties concerning the k-step Fibonacci numbers are obtained and related to the spectral radius of the k-Fibonacci matrices in [3], the powers of the k-Fibonacci matrices are investigated and closed formulas for their entries are derived, related to the suitable terms of the $k-$step Fibonacci sequences as well as the properties of the irreducibility and the primitivity of the associated k-Fibonacci matrices are discussed in [1].

In the present paper, the powers of the generalized 2-Fibonacci matrices are investigated and closed formulas for their entries are derived, which are related to the combinatorial representation of the nonnegative constant real numbers c_1, c_2 defined the associated generalized 2-step Fibonacci sequence.

2 Generalized k-step Fibonacci sequences and matrices

In [2], for the integer $k = 1, 2, \ldots$, and the nonnegative constant real numbers $c_1, c_2, \ldots, c_k$, where $c_1 > 0$, the n-th term f_n of the *generalized k-step*

Fibonacci sequence, $(f_n(c_1, c_2, \ldots, c_k))_{n=1,2,\ldots}$, is defined by the recursive formulation

$$\begin{aligned} f_n &= c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k} \\ &= \sum_{j=1}^{k} c_j f_{n-j}, \quad \text{for every } \ n \geq k+1, \end{aligned} \tag{1}$$

with

$$f_1 = f_2 = \ldots = f_k = 1. \tag{2}$$

From $c_1 > 0$, $c_2, c_3, \ldots, c_k \geq 0$ and (1)-(2), it is obvious that all the terms f_n of the generalized k-step Fibonacci sequence $(f_n(c_1, c_2, \ldots, c_k))_{n=1,2,\ldots}$ are positive real numbers.

Remark 2.1 **(i)** From (1)-(2) it is evident that for $k = 1$, the n-th term f_n of the associated generalized Fibonacci sequence $(f_n(c_1))_{n=1,2,\ldots}$ is equal to $f_n = c_1^{n-1}$, $c_1 > 0$. Hereafter consider $k \geq 2$, since the case $k = 1$ is trivial.

(ii) Moreover, notice that for $k \geq 2$, and $c_1 > 0$, $c_2 = c_3 = \ldots = c_k = 0$, the n-th term f_n of the generalized Fibonacci sequence $(f_n(c_1, 0, \ldots, 0))_{n=1,2,\ldots}$ is equal to $f_n = c_1^{n-k}$, $c_1 > 0$. Hereafter consider at least two nonzero coefficients c_i, $i = 1, 2, \ldots, k$, in (1) because otherwise we have a trivial case.

(iii) Using $m = 0$, the equations (1)-(2) are derived immediately by the definition of the generalized k, m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$, [2].

(iv) For $c_1 = c_2 = \ldots = c_k = 1$, the generalized k-step Fibonacci sequence $(f_n(1, 1, \ldots, 1))_{n=1,2,\ldots}$ gives well-known sequences for various values of k. In particular,
-for $k = 2$, the equations (1)-(2) give the well-known Fibonacci sequence, $1, 1, 2, 3, 5, 8, 13, \ldots$, [1, 3, 12].
-for $k = 3$, (1)-(2) give the tribonacci sequence, $1, 1, 1, 3, 5, 9, 17, \ldots$, [3, Remark 2(ii)].
-for $k = 4$, (1)-(2) give the tetranacci sequence, $1, 1, 1, 1, 4, 7, 13, \ldots$, [3, Remark 2(iii)].

The *generalized k-Fibonacci matrix* has been first introduced in [8] and it is defined as the nonnegative $k \times k$ matrix

$$Q_k(c_1, c_2, \ldots, c_k) = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \tag{3}$$

where the first row of the above matrix has entries the nonnegative real numbers $c_1 > 0$, $c_2, c_3, \ldots, c_k \geq 0$, for $k \geq 2$.

Remark 2.2 **(i)** In [2, 6, 9, 10], the determinant of the generalized k-Fibonacci matrix in (3) is formulated by

$$det(Q_k(c_1, c_2, \ldots, c_k)) = (-1)^{k+1} c_k, \tag{4}$$

and the $k-$th degree characteristic polynomial $x_{Q_k(c_1,c_2,\ldots,c_k)}(\lambda)$ of the generalized $k-$Fibonacci matrix $Q_k(c_1, c_2, \ldots, c_k)$ has been proved in [8] and it is given by

$$x_{Q_k(c_1,c_2,\ldots,c_k)}(\lambda) = \lambda^k - \sum_{i=1}^{k} c_i \, \lambda^{k-i}. \tag{5}$$

(ii) From (4) it is obvious that $Q_k(c_1, c_2, \ldots, c_k)$ is a nonsingular matrix if and only if $c_k \neq 0$, and then all the eigenvalues of $Q_k(c_1, c_2, \ldots, c_k)$ are nonzero.

(iii) The trace of a matrix A is denoted by $tr(A)$. From (3) it is evident that

$$tr(Q_k(c_1, c_2, \ldots, c_k)) = c_1.$$

(iv) For $c_1 = c_2 = \ldots = c_k = 1$, the relationships between the Fibonacci numbers and their associated k-Fibonacci matrices $Q_k(1, 1, \ldots, 1)$ and the powers of $Q_k(1, 1, \ldots, 1)$ have been discussed in [1, 3, 7, 14], as well as the properties of the irreducibility and primitivity of $Q_k(1, 1, \ldots, 1)$ have been investigated in [1].

(v) The generalized 2-Fibonacci matrix is defined by (3) for $k = 2$, $c_1 > 0$, $c_2 \geq 0$ and in the following it is formulated as the nonnegative 2×2 matrix

$$Q_2(c_1, c_2) = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}. \tag{6}$$

In the following theorem, the Pascal 's triangle identity is needed, which is formulated as

$$\begin{pmatrix} n-\tau \\ \tau \end{pmatrix} + \begin{pmatrix} n-\tau \\ \tau-1 \end{pmatrix} = \begin{pmatrix} n-\tau+1 \\ \tau \end{pmatrix}. \tag{7}$$

Theorem 2.3 *Let the positive real numbers c_1, c_2 and the associated generalized $2-$Fibonacci matrix $Q_2(c_1, c_2)$ in (6). Let $n \geq 2$, then the n power of $Q_2(c_1, c_2)$ is defined as*

$$Q_2^n(c_1, c_2) = (Q_2(c_1, c_2))^n = (Q_2(c_1, c_2))^{n-1} Q_2(c_1, c_2) = \begin{bmatrix} q_{11}^{(n)} & q_{12}^{(n)} \\ & \\ q_{21}^{(n)} & q_{22}^{(n)} \end{bmatrix}, \tag{8}$$

where the positive real numbers $q_{11}^{(n)}, q_{12}^{(n)}, q_{21}^{(n)}, q_{22}^{(n)}$ are given by

$$q_{11}^{(n)} = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-r \\ r \end{pmatrix} c_1^{n-2r} c_2^r, \tag{9}$$

$$q_{12}^{(n)} = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-r \\ r \end{pmatrix} c_1^{n-1-2r} c_2^{r+1}, \tag{10}$$

$$q_{21}^{(n)} = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-r \\ r \end{pmatrix} c_1^{n-1-2r} c_2^{r}, \tag{11}$$

$$q_{22}^{(n)} = \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} \begin{pmatrix} n-2-r \\ r \end{pmatrix} c_1^{n-2-2r} c_2^{r+1}, \tag{12}$$

and $\lfloor n \rfloor$ denotes the floor function of n.

Proof The proof of (9)-(12) is based on the induction method on n.

For $n = 2$, the entries of matrix in (8) are trivially verified by the formulas in (9)-(12), since holds

$$Q_2^2(c_1,c_2) = (Q_2(c_1,c_2))^2 = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} c_1^2 + c_2 & c_1 c_2 \\ c_1 & c_2 \end{bmatrix} = \begin{bmatrix} q_{11}^{(2)} & q_{12}^{(2)} \\ q_{21}^{(2)} & q_{22}^{(2)} \end{bmatrix}. \quad (13)$$

Notice that combining (6) and (8), the $(n+1)$ power of $Q_2(c_1,c_2)$ is formulated by

$$Q_2^{n+1}(c_1,c_2) = Q_2^n(c_1,c_2)Q_2(c_1,c_2) = \begin{bmatrix} q_{11}^{(n)} & q_{12}^{(n)} \\ q_{21}^{(n)} & q_{22}^{(n)} \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c_1 q_{11}^{(n)} + q_{12}^{(n)} & c_2 q_{11}^{(n)} \\ c_1 q_{21}^{(n)} + q_{22}^{(n)} & c_2 q_{21}^{(n)} \end{bmatrix}. \quad (14)$$

Consider that n is an arbitrary even positive number less than 2 and assume that the formulas in (9)-(10) are true for $n = 2m$, $(m \in \mathbb{N})$, by (14) the $q_{11}^{(n+1)}$ entry of $Q_2^{n+1}(c_1,c_2)$ is formulated as

$$\begin{aligned}
q_{11}^{(n+1)} &= c_1 q_{11}^{(n)} + q_{12}^{(n)} \\
&= c_1 \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r} c_1^{n-2r} c_2^r + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} c_1^{n-1-2r} c_2^{r+1} \\
&= c_1 \sum_{r=0}^{m} \binom{n-r}{r} c_1^{n-2r} c_2^r + \sum_{r=0}^{m-1} \binom{n-1-r}{r} c_1^{n-1-2r} c_2^{r+1} \\
&= c_1^{n+1} + \sum_{r=1}^{m} \binom{n-r}{r} c_1^{n+1-2r} c_2^r + \sum_{\tau=1}^{m} \binom{n-\tau}{\tau-1} c_1^{n-1-2(\tau-1)} c_2^{\tau} \\
&= c_1^{n+1} + \sum_{\tau=1}^{m} \binom{n-\tau}{\tau} c_1^{n+1-2\tau} c_2^{\tau} + \sum_{\tau=1}^{m} \binom{n-\tau}{\tau-1} c_1^{n+1-2\tau} c_2^{\tau} \\
&= c_1^{n+1} + \sum_{\tau=1}^{m} \{ \binom{n-\tau}{\tau} + \binom{n-\tau}{\tau-1} \} c_1^{n+1-2\tau} c_2^{\tau}.
\end{aligned}$$

Using (7) in the above equality, it is formulated as

$$\begin{aligned}
q_{11}^{(n+1)} &= c_1^{n+1} + \sum_{\tau=1}^{m} \binom{n-\tau+1}{\tau} c_1^{n+1-2\tau}\, c_2^{\tau} \\
&= \sum_{\tau=0}^{m} \binom{n+1-\tau}{\tau} c_1^{n+1-2\tau}\, c_2^{\tau} \\
&= \sum_{\tau=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1-\tau}{\tau} c_1^{n+1-2\tau}\, c_2^{\tau} = \sum_{\tau=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-\tau}{\tau} c_1^{n+1-2\tau}\, c_2^{\tau},
\end{aligned}$$

since n is an even number. Hence, (9) holds also for odd number $n+1$, which completes the induction method for the formula of $q_{11}^{(n)}$. Similarly, it is proved the case for $n = 2m+1,\ m \in \mathbb{N}$.

Moreover, assuming that the formulas in (11)-(12) are true for $n = 2\nu+1$, $\nu \in \mathbb{N}$, and using analogous statements as in the proof of $q_{11}^{(n)}$, by (14) the $q_{21}^{(n+1)}$ entry of $Q_2^{n+1}(c_1, c_2)$ is given by

$$\begin{aligned}
q_{21}^{(n+1)} &= c_1 q_{21}^{(n)} + q_{22}^{(n)} \\
&= c_1 \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n-1-r \\ r \end{pmatrix} c_1^{n-1-2r} c_2^r + \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} \begin{pmatrix} n-2-r \\ r \end{pmatrix} c_1^{n-2-2r} c_2^{r+1} \\
&= \sum_{r=0}^{\nu} \begin{pmatrix} n-1-r \\ r \end{pmatrix} c_1^{n-2r} c_2^r + \sum_{r=0}^{\nu-1} \begin{pmatrix} n-2-r \\ r \end{pmatrix} c_1^{n-2-2r} c_2^{r+1} \\
&= c_1^n + \sum_{r=1}^{\nu} \begin{pmatrix} n-1-r \\ r \end{pmatrix} c_1^{n-2r} c_2^r + \sum_{\tau=1}^{\nu} \begin{pmatrix} n-2-(\tau-1) \\ \tau-1 \end{pmatrix} c_1^{n-2-2(\tau-1)} c_2^{\tau} \\
&= c_1^n + \sum_{\tau=1}^{\nu} \begin{pmatrix} n-1-\tau \\ \tau \end{pmatrix} c_1^{n-2\tau} c_2^{\tau} + \sum_{\tau=1}^{\nu} \begin{pmatrix} n-1-\tau \\ \tau-1 \end{pmatrix} c_1^{n-2\tau} c_2^{\tau} \\
&= c_1^n + \sum_{\tau=1}^{\nu} \{ \begin{pmatrix} n-1-\tau \\ \tau \end{pmatrix} + \begin{pmatrix} n-1-\tau \\ \tau-1 \end{pmatrix} \} c_1^{n-2\tau} c_2^{\tau}. \qquad (15)
\end{aligned}$$

Using the Pascal 's identity by (7) the equality in (15) can be written as

$$\begin{aligned}
q_{21}^{(n+1)} &= c_1^n + \sum_{\tau=1}^{\nu} \begin{pmatrix} n-\tau \\ \tau \end{pmatrix} c_1^{n-2\tau}\, c_2^{\tau} \\
&= \sum_{\tau=0}^{\nu} \begin{pmatrix} n-\tau \\ \tau \end{pmatrix} c_1^{n-2\tau}\, c_2^{\tau} \\
&= \sum_{\tau=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-\tau \\ \tau \end{pmatrix} c_1^{n-2\tau}\, c_2^{\tau}.
\end{aligned}$$

Hence, (11) holds also for odd number $n+1$, which completes the induction method for the formula of $q_{21}^{(n)}$. Similarly, it is proved the case for $n = 2\nu,\ \nu \in \mathbb{N}$.

From (14) it is obvious that multiplying the first column of $Q_2^n(c_1, c_2)$ with c_2 arises the second column of $Q_2^{n+1}(c_1, c_2)$; hence, using the formulas in (9) and (11), the associated entries of the second column of $Q_2^{n+1}(c_1, c_2)$ are given, which completes the induction method for (10) and (12), respectively. $\square$

Remark 2.4 **(i)** Consider the special case $c_1 = c_2 = 1$ in the formulas (9)-(12), then the entries of $Q_2^n(1,1)$ in (8) are formulated as in Theorem 2.3 and the matrix $Q_2^n(1,1)$ is given by

$$Q_2^n(1,1) = \begin{bmatrix} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r} & \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} \\ \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} & \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-r}{r} \end{bmatrix}. \qquad (16)$$

Moreover, the entries of $Q_2^n(1,1)$ can be related to the suitable terms of the 2-step Fibonacci sequence and the associated formulas have been proved in [1, Theorem 3.4]. In particular, in [1, Remark 3.1(iii)] the formula of $Q_2^n(1,1)$ has been given as

$$Q_2^n(1,1) = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}; \qquad (17)$$

recall that f_{n+1}, f_n, f_{n-1} denote the Fibonacci numbers for $n \geq 2$, which are implied by (1)-(2) for $c_1 = c_2 = 1$.

Combining the associated formulas in (16) and (17) all the terms of the well-known 2-Fibonacci sequence in Remark 2.1 (iv) can be expressed as a sum of suitable binomial coefficients as following;

$$f_{n+1} = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r}, \quad \text{for } n \geq 2$$

(ii) The general idea of the Fibonacci cryptography is based on the matrix $Q_2^n(1,1)$ in (17) of the above Remark 2.4(i), (see, the associated metodology in [16, 17]). Now, using in the process of the cryptography of an initial message the generalized 2-Fibonacci matrix $Q_2^n(c_1,c_2)$ in (8) for the arbitrary $c_1, c_2 > 0$, one can provide higher security for encryption and decryption, since $Q_2^n(c_1,c_2)$ is a nonsingular matrix (see, in the above Remark 2.2 (ii)) and the closed formulas in (9)-(12) for the entries of $Q_2^n(c_1,c_2)$ can be computed a-priori for various values of c_1, c_2.

References

[1] M. Adam and N. Assimakis, $k-$step Fibonacci sequences and Fibonacci matrices, *Journal of Discrete Mathematical Sciences & Cryptography*, (2016), (in press).

[2] M. Adam, N. Assimakis and G. Tziallas, Generalized k, m-step Fibonacci sequences and matrices, *Proceedings of the 12th International Conference of AHA-Algebraic Hyperstructures and its Applications in Int. J. of Algebraic Hyperstructures and Its Applications*, **2**(1), (2015), 125-134.

[3] M. Adam and N. Assimakis, k-step sum and m-step gap Fibonacci sequence, *ISRN Discrete Mathematics*, v. 2014, Article ID 374902, (2014), 7 pages, http://dx.doi.org/10.1155/2014/374902.

[4] M.C. Er, The matrices of Fibonacci numbers, *The Fibonacci Quarterly*, **22**(2), (May, 1984), 134-139.

[5] M.C. Er, Sums of Fibonacci numbers by matrix methods, *The Fibonacci Quarterly*, **22**(3), (1984), 204-207.

[6] Xudan Fu and Xia Zhou, On matrices related with Fibonacci and Lucas numbers, *Applied Mathematics and Computation*, **200**, (2008), 96-100.

[7] John Ivie, A general Q-matrix, *The Fibonacci Quarterly*, **10**(3), (1972), 255-264.

[8] Dan Kalman, Generalized Fibonacci numbers by matrix methods, *The Fibonacci Quarterly*, **20**(1), (1982), 73-76.

[9] Erdal Karaduman, *On determinants of matrices with general Fibonacci numbers entries*, Applied Mathematics and Computation, **167** (2005), 670-676.

[10] Erdal Karaduman, An application of Fibonacci numbers in matrices, *Applied Mathematics and Computation*, **147**, (2004), 903-908.

[11] Emrah Kilic, The generalized order-k Fibonacci-Pell sequence by matrix methods, *Journal of Computational and Applied Mathematics* , **209**, (2007), 133-145.

[12] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., New York, 2001.

[13] G.-Y. Lee, J-S. Kim, S.-H. Cho, Some combinatorial identities via Fibonacci numbers, *Discrete Applied Mathematics*, **130**, (2003), 527 - 534.

[14] G.-Y. Lee and S.-G. Lee, A note on generalized Fibonacci numbers, *The Fibonacci Quarterly*, **33**(3), (1995), 273-277.

[15] G.-Y. Lee, S.-G. Lee and H.-G. Shin, On the k-Generalized Fibonacci matrix Q_k, *Linear Algebra and its Applications*, **251**, (1997), 73-88.

[16] M. Mishra, P. Mishra, M.C. Adhikary and S. Kumar, Image encryption using Fibonacci-Lucas transformation, *International Journal on Cryptography and Information Security*(IJCIS), **2**(3), (September, 2012), 131-141.

[17] L.-P. Shao, Z. Qin, H.-L.Gao and X.-C. Heng, 2D triangular mappings and their applications in scrambling rectangle image, *Information Technology Journal*, **7**(1), (2008), 40-47.

[18] Sheng-liang Yang, On the k-generalized Fibonacci numbers and high-order linear recurrence relations, *Applied Mathematics and Computation*, **196**, (2008), 850-857.