

## **Revisiting Uncertain Investment and Financing Decisions using Entry Probability**

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### **Abstract**

This paper extends the work of Chen and Chang (2010) and attempts to present a model for the optimal investment threshold and the real option value under price uncertainty from a different aspect of entry probability. I measure a financing policy by the debt ratio, a weight for the proportion of funding by debt relative to funding by equity. The weight is exogenously embedded in the stochastic optimization of an investment opportunity under price uncertainty. An entry probability, indicating the likelihood of an investment action optimally taken at a future time instant, is derived. Sensitivity analyses are performed to investigate the effects of parameters such as price volatility and price drift. The results demonstrate that price drift and price volatility might have different effects, depending on the time elapsed and on the debt ratio. In general, higher debt ratio contributes to higher investment threshold and higher real option value, confirming that financial leverage might serve as a value driver. However, as the debt ratio increases, the investment threshold might become too high to reach, indicating that waiting is the best choice. Such a circumstance is also verified by the entry probability approaching zero, which implies the far-reaching investment threshold.

**JEL classification numbers:** C61, G11, G32

**Keywords:** Price uncertainty, Financing policy, Investment threshold, Real option, Entry probability

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## 1 Introduction

As the product life cycle in a highly competitive market of consumer electronics becomes shorter and shorter, the launch of a new product as an investment opportunity faces more and more risks than before. The overall risk in an investment opportunity can be measured by both operational and financial leverage. The operational leverage relates to the market uncertainty such as price volatility in a product market, whereas the financial leverage relates to financing uncertainty such as variable interest payments in a capital market. An investment, financed partly from the stock market and partly from the debt market, is the one with financial leverage. Even though the interest payments in funding by debt increase the expenditures, they also decrease the tax payables. Such a tradeoff is referred to as the effect of tax shield. In other words, the interest payments have two opposing effects on project value: increasing expenditures and saving the cash outflows of taxes. Besides, tax exemption or tax incentive may be provided by governments in many countries to boost the development of an industry, such as the semiconductor industry in Taiwan.

Modigliani and Miller [1] advocated the irrelevance of investment and financing decisions in the absence of tax and bankruptcy cost. However, the further development of their theory releases the tax-free and bankruptcy-cost-free assumptions and then switches from independence to interdependence between investment and financing decisions (Brennan and Schwartz [2], Mellon and Parsons [3], Mauer and Triantis [4], Mauer and Ott [5], Hennessy and Whited [6], and Titman and Tsyplakov [7]). The track advocating relevance between investment and financing decisions becomes more and more popular. For example, Strebulaev [8] further distinguished funding policy between external debt and internal equity sources. Among these scholars who focus on interdependence between investment and financial decisions, Titman and Tsyplakov lays emphasis on firms' investment flexibility, allowing a firm to undertake any investment project at the optimal entry time. The study of Azofra and Miguel [9] empirically verified a direct relationship between investment and debt. The empirical study of Su and Vo [10] found that combined effect of corporate strategy and capital structure explains firms' performance very well. In contrast, I intend to propose an optimization model to deal with the relationship between investment timing and funding policy from a different aspect of entry probability. Regarding the applications of entry probability, Kamien and Schwartz [11] used entry probability to investigate the effect of pricing policy on the entry probability of competing suppliers. Besides, Mitchell [12] examined the entry probability and entry timing for industrial incumbents preparing to enter the corresponding technical subfields. In contrast, I employ the entry probability to revisit the relevance between the investment timing and the financing policy of capital structure.

Unlike most applications of optimization intending to solve for tradeoffs between two opposing forces, this work adopts stochastic optimization in time domain. For example, Chen [13] considers two opposing forces: quality loss and inventory cost in the optimization of the expected total cost to obtain the optimal production run length and process mean. Contrarily, I consider a price process with stochastic nature to evolve. The stochastic price comprises stochastic profit flow, which in turn comprises the value process of an investment. As time passes, the value process might evolve upwards or downwards in the future. There must exist a certain time instant in the future when maximum expected present value occurs. To obtain the optimal value in the stochastic context, stochastic optimization is therefore employed by this work.

In this work, I follow the same derivation procedures as Chen and Chang [14] proposed to investigate the impact of financing policy on investment strategy under uncertainty. The investment project value is derived by discounting all the future profits, which evolve as a stochastic process due to the assumption of randomness in price. Based on the historical price quotes for the LED components from September 1999 to June 2009 acquired from the database of Taiwan Economic Journal (TEJ), an example is illustrated in figure 1 to demonstrate the stochastic price process. The figure also demonstrates that the quantity demanded is negatively correlated with the corresponding price. Such phenomenon is modeled by the inverse demand function in this model.

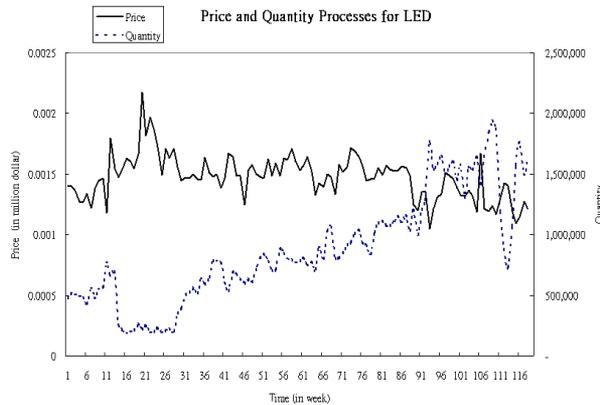


Figure 1: The monthly price process denominated in the US dollar for LED components.

This paper is organized as follows: Section 1 introduces the previous researches on capital structure in corporate finance. Section 2 presents the formulation of project value in the context of stochastic price under an inverse demand function. The analytical solutions for investment threshold, real option value and corresponding entry probability are then derived. In Section 3, I present the analytical results in figures to analyze the sensitivity. Section 4 concludes this paper.

## 2 The Framework

### 2.1 Notation and Modeling

In this model, following the framework of Chen and Chang [14], I calculate the expected present value of future cash flows generated from an investment project facing the price uncertainty. Because of the stochastic effects in the price, the uncertainty will be propagated into the future cash flows, evolving stochastically on the continuous time basis, and then into the discounted value, whose expectation represents the project value in our model. It is assumed that the price is a stochastic process and denoted by  $p_t$ , which evolves as a geometric Brownian motion as shown in equation (1).

$$dp_t = \mu p_t dt + \sigma p_t dZ_t, \tag{1}$$

where  $\mu$  is a constant price drift,  $\sigma$  is a constant price volatility, and  $Z_t$  is a standard Brownian motion with a filtration  $\mathfrak{F}^p(t), t \geq 0$ ; thus  $dZ_t$  is the normally distributed increment with the expected value 0 and the variance  $dt$ .

I assume that the market demand is price sensitive and can be defined by an inverse demand function of Carruth et al. [15] as follows.

$$P_t = Q_t^{-\frac{1}{\varepsilon}}. \quad (2)$$

The firm's marginal cost is assumed to be proportional to its revenue as the assumption of Schwartz and Moon [16], who divided the cost structure into two components: a fixed term and a variable term which is proportional to the revenue. I also assume that the firm could control its cost rate at a fixed level relative to the revenues through its managerial activities such as its bargaining power to negotiate with the supplier. I denote the marginal cost incurred at time  $t$  as  $v_t$ , which is defined as follows.

$$v_t = \gamma \cdot P_t, \quad (3)$$

where  $\gamma$  is a constant cost rate.

It is assumed that the market share of the firm is constant and denoted by  $m$ . I denote the revenue at time  $t$  as  $R_t$  and the total cost as  $C_t$ , which can be expressed as follows.

$$R_t = P_t m Q_t, \quad (4)$$

and

$$C_t = v_t m Q_t. \quad (5)$$

I assume that total capital demand for the firm is proportional to its revenues, and denoted by  $K_t$ . Then I have

$$K_t = b R_t, \quad (6)$$

where  $b > 0$ .

I assume that the capital demand of the project is financed through issuing debt and equity with weights  $w$  and  $1 - w$ , respectively. The weight  $w$  is what Kim et al. [17] used to measure the capital structure, and referred to as the debt ratio, which is defined as the ratio of debt to total capital demand. According to business practice, I set  $0 \leq w \leq 1$ . When  $w = 0$ , the investment project is all-equity financed without financial leverage. In contrast, when  $w = 1$ , the investment project is an all-debt one without any stock issued. I denote the debt and equity amounts as  $D_t$  and  $E_t$ . The following two equations describe the relationships between revenue and debt amount, as well as equity amount.

$$D_t = w K_t, \quad (7)$$

and

$$E_t = (1 - w) K_t. \quad (8)$$

I denote the funding rates from both creditors and shareholders as  $r_D$  and  $r_E$ . The tax rate is denoted by  $\eta$ . As proposed by Lin [18], the weighted average of the cost of capital (WACC) for the firm is employed as the discount rate, which is denoted by  $\delta$  and defined as follows:

$$\delta = (r_D(1 - \eta) - r_E)w + r_E. \quad (9)$$

The interest payment to the creditors is  $D_t r_D$ . I then denote the profit flow after tax as  $\pi_t$ , which can be formulated as follows:

$$\pi_t = (R_t - C_t - D_t r_D)(1 - \eta). \quad (10)$$

The funding rate from creditors has two opposing effects on the project value: a negative effect on the profit flow and a positive effect on the discount rate. In practice, shareholders require a higher rate of return for their investments because of their assuming higher risk than creditors. I have  $r_D < r_E$ . For the interest payment, higher  $w$  leads to higher debt amount and thus higher interest payment, as well as lower profit flow. In contrast, according to equation (9), higher  $w$  leads to lower discount rate and thus higher present value of any future profit flow.

I assume that the firm obtains sustainable growth in its investment project through continuous innovation. Consequently, the future profit flows last indefinitely. The investment project value under price uncertainty can be obtained by the same procedures as Chen and Chang [20] derived:

$$V_t = \mathbf{E} \left[ \int_t^\infty \pi_s \cdot e^{-\delta(s-t)} ds \mid \mathfrak{F}^p(t)_t \right] = \frac{(1 - \gamma - w b r_D)(1 - \eta)m}{\delta - (\mu - \frac{1}{2} \sigma^2 \varepsilon)(1 - \varepsilon)} P_t^{1-\varepsilon} \quad (11)$$

where

$$\delta - (\mu - \frac{1}{2} \sigma^2 \varepsilon)(1 - \varepsilon) > 0.$$

## 2.2 Derivation of Analytical Solution

By Ito's lemma, I can expand the increment of the investment project value at time  $t$  from equation (11) as follows.

$$\frac{dV_t}{V_t} = (1 - \varepsilon)(\mu dt + \sigma dW_t) + \frac{1}{2}(1 - \varepsilon)(-\varepsilon)\sigma^2 dt. \quad (12)$$

Rewrite equation (12), I can derive the dynamics of the investment project value as follows.

$$\frac{dV_t}{V_t} = u dt + v dZ_t,$$

where

$$u = (1 - \varepsilon)(\mu - \frac{1}{2} \varepsilon \sigma^2) \text{ and } v = (1 - \varepsilon)\sigma. \quad (13)$$

Subsequently, the firm could determine the optimal entry time for the investment. It should pay the irreversible sunk cost ( $I$ ) for the installation of the operation facilities and then acquire the project value at time  $s$  ( $V_s$ ) in return. The investment profit at time  $s$  is  $V_s - I$ . Therefore, the optimization for the investment could be described as equation (14). The objective function, referred to as the real option value, is denoted by  $F(V)$  as follows:

$$F(V) = \max_{s>t} \mathbf{E}_0 \left[ (V_s - I) e^{-\delta s} \mid \mathfrak{F}^p(t) \right]. \quad (14)$$

The optimal entry time for the investment is the corresponding time when the optimization is achieved. To optimize equation (14), the corresponding Bellman equation should be employed.

$$\delta F dt = E(dF | \mathfrak{F}^p(t)). \quad (15)$$

The expected capital appreciation can be expressed as follows:

$$E(dF | \mathfrak{F}^p(t)) = \left\{ u V_t F_V + \frac{1}{2} v^2 V_t^2 F_{VV} \right\} dt. \quad (16)$$

Substituting equation (16) into (15), I can obtain the differential equation as follows:

$$u V_t F_V + \frac{1}{2} v^2 V_t^2 F_{VV} - \delta F = 0. \quad (17)$$

Solving for Equation (17), I can finally obtain the option value  $F(V_t)$ .

Assume that the option value is in the following form:

$$F(V_t) = c_1 \cdot (V_t)^{d_1} + c_2 \cdot (V_t)^{d_2} \quad (18)$$

where  $c_1, c_2$  are two constant coefficients.

Equation (18) is reduced to the following equation:

$$F(V_t) = c_1 (V_t)^{d_1}. \quad (19)$$

I denote the value threshold as  $V^*$ . The other two boundary conditions are as follows:

$$F(V^*) = V^* - I, \quad (20)$$

and

$$F_V(V^*) = 1. \quad (21)$$

According to the work of Dixit and Pindyck [19], I can derive coefficients  $d_1, c_1$  and thresholds  $V^*$  from Equations (19), (20) and (21) as follows:

$$c_1 = \left( \frac{I}{d_1 - 1} \right)^{1-d_1} \cdot d_1^{-d_1} \quad (22)$$

and

$$V^* = \left( \frac{d_1}{d_1 - 1} \right) \cdot I \quad (23)$$

where

$$d_1 = \left( \frac{1}{2} - \frac{u}{v^2} \right) + \sqrt{\left( \frac{1}{2} - \frac{u}{v^2} \right)^2 + \frac{2\delta}{v^2}},$$

$$u = (1 - \varepsilon)\sigma,$$

$$v = \left( \mu - \frac{1}{2} \varepsilon \sigma^2 \right) (1 - \varepsilon),$$

and

$$\delta = (r_D(1 - \eta) - r_E)w + r_E.$$

Substituting equation (11) into (19), I can finally obtain the real option value as follows.

$$F(V_t) = c_1 \left( \frac{(1 - \gamma - wbr_D)(1 - \eta)m}{\delta - (\mu - \frac{1}{2}\sigma^2\varepsilon)(1 - \varepsilon)} P_t^{1-\varepsilon} \right)^{d_1} \tag{24}$$

From equations (23) and (24), it can be shown that the price drift and volatility, the debt ratio, cost and tax rates, and the price elasticity of demand jointly determine not only the investment threshold but also the real option value. Consequently, the firm should intensively observe those factors for its optimal entry decision.

In order to understand the likelihood of entry for the investment opportunity, I derive the entry probability shown as equation (25), where  $N(k)$  is a standard normal distribution function.

$$\begin{aligned} & \text{prob}(V_s \geq V^*) \\ &= \text{prob}(V_t e^{(u-0.5v^2)(s-t)+v\sqrt{s-t}\varepsilon_t} \geq V^*) \\ &= \text{prob}(\varepsilon_t \geq \frac{\ln \frac{V^*}{V_t} - (u - 0.5v^2)(s-t)}{v\sqrt{s-t}}) \end{aligned} \tag{25}$$

$$= 1 - N(k)$$

$$\text{where } k = \frac{\ln \frac{V^*}{V_t} - (u - 0.5v^2)(s-t)}{v\sqrt{s-t}}.$$

In order to have a more complete understanding of the effect of debt ratio on the entry probability, I derive equation (26) (proved in Appendix) to demonstrate the sensitivity of entry probability to debt ratio.

$$\begin{aligned} & \frac{\partial \text{prob}(V_s \geq V^*)}{\partial w} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{I}{m(1-\eta)P_t^{1-\varepsilon}} \\ & \quad \cdot \left( \frac{\beta \left( \left( \frac{d_1}{d_1-1} \right) + \left( \frac{h^{-0.5}}{v^2(d_1-1)^2} \right) (\delta - v) \right)}{\alpha} + \frac{(br_D)(\delta - v)}{\alpha^2} \right) \end{aligned} \tag{26}$$

where

$$h = \left( \frac{1}{2} - \frac{u}{v^2} \right)^2 + \frac{2\delta}{v^2},$$

$$k = \frac{\ln \frac{V^*}{V_t} - (u - 0.5v^2)(s - t)}{v\sqrt{s - t}},$$

$$\alpha = 1 - \gamma - wbr_D,$$

and

$$\beta = r_D(1 - \eta) - r_E.$$

Similarly, equation (27) can be derived to demonstrate the sensitivity of entry probability to time index  $s$  ( $s > t$ ).

$$\frac{\partial \text{prob}(V_s \geq V^*)}{\partial s} = \frac{1}{2v\sqrt{2\pi}} \cdot \frac{u - 0.5v^2}{\sqrt{s - t}} \cdot e^{-\frac{k^2}{2}} \tag{27}$$

### 3 Sensitivity Analysis

In this section, I shall discuss the effects of parameters on the threshold and the real option value. Based on equations (23) and (24), sensitivity analyses are performed and illustrated in figures 3 through 14 by the same parameter set as Chen and Chang [14] adopted:  $\mu = 0.2, \sigma = 0.3, m = 0.8, r_D = 4\%, r_E = 5\%, \eta = 25\%, \gamma = 0.4$  and  $\varepsilon = 0.9$ .

In figure 2, as Chen and Chang [14] discussed, the investment threshold slightly increases with increasing price drift, but the curves with different price drift are almost overlapped, indicating that price drift does not affect the entry threshold. In figure 3, it is shown that the higher the price volatility, the higher the threshold. The debt ratio strengthens the increasing effect of price volatility on threshold, causing the investment threshold to rise sharply. The sharp rise of entry threshold may cause the entry less possible. This phenomenon could be explained later in terms of entry probability.

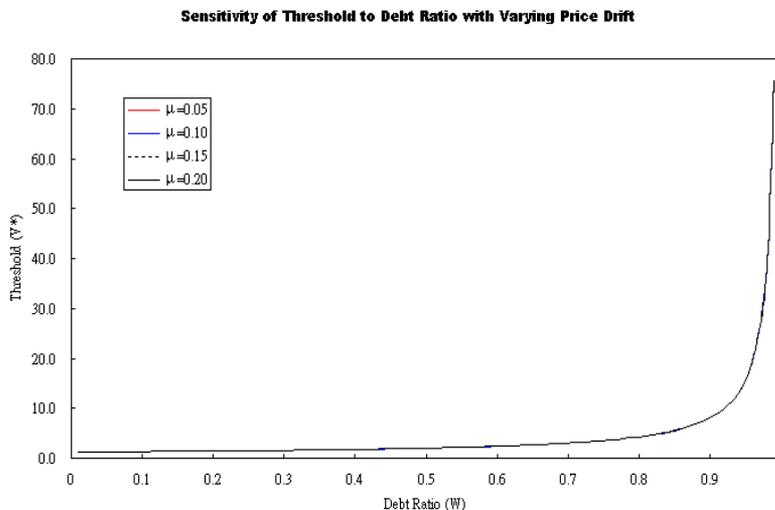


Figure 2: The sensitivity of investment threshold to debt ratio under varying price drift

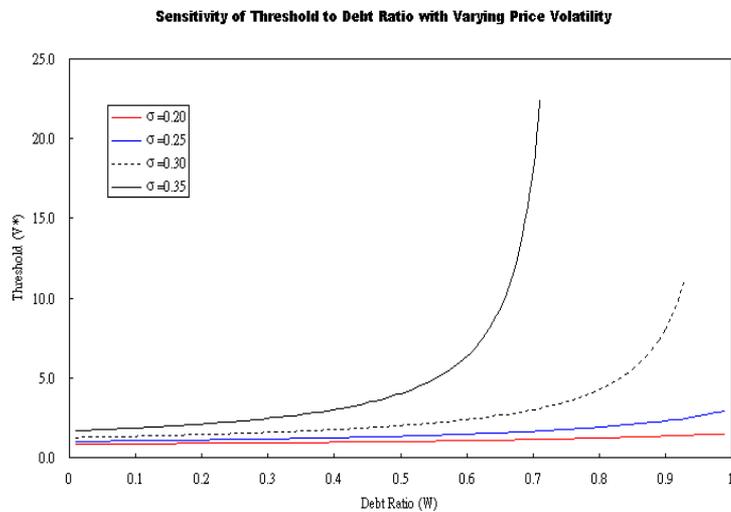


Figure 3: The sensitivity of investment threshold to debt ratio under varying price volatility

As Chen and Chang [14] discussed, figure 4 shows that the higher the price drift, the higher the real option value, while higher debt ratio raises the real option value more sharply. On the contrary, in figure 5, the higher the price volatility, the lower the real option value. However, the impact of debt ratio is not as significant as in figure 9.

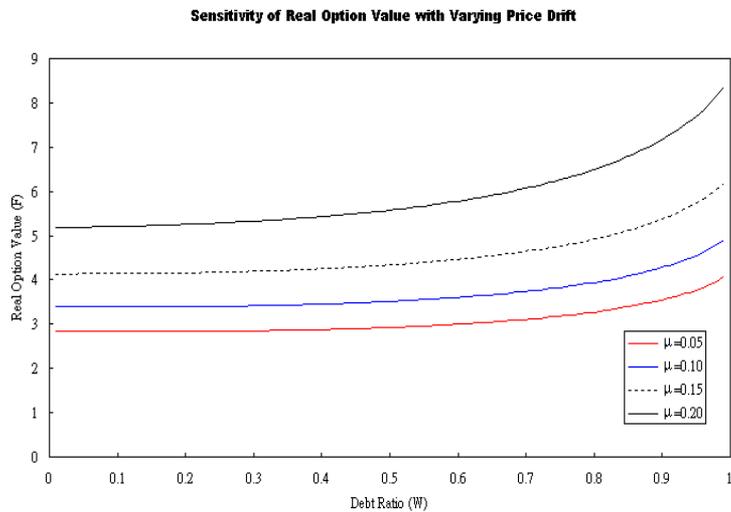


Figure 4: The sensitivity of real option value to debt ratio under varying price drift

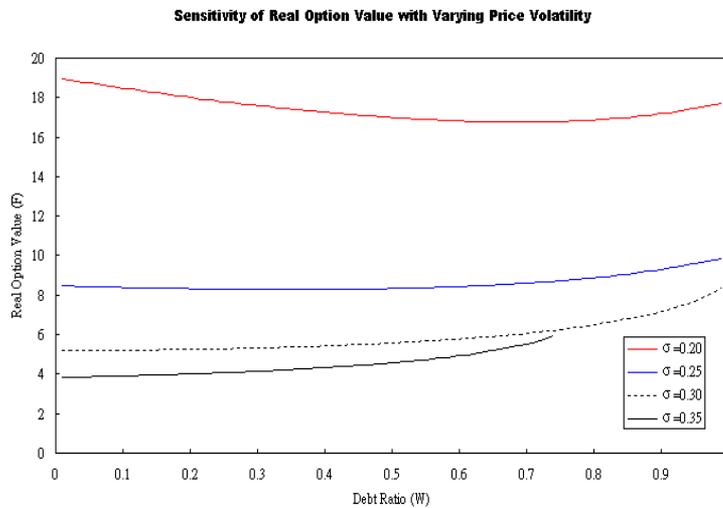


Figure 5: The sensitivity of real option value to debt ratio under varying price volatility

The investment activity is activated when the stochastic investment project value goes up through the threshold. Therefore, the higher the threshold, the less likely the investment is undertaken. Based on equation (25), one example is illustrated in figure 6 to demonstrate the effects of debt ratio and time duration on the entry probability. For example, when debt ratio is 0.1, the optimal entry time may be later than year 6 with entry probability 1.0. To have a more complete understanding of the effects of debt ratio and time duration on the entry probability, equation (26) and (27) are derived to demonstrate the sensitivity of entry probability to debt ratio, and to time duration. It is shown in figure 7 that the change rate of entry probability with respect to debt ratio is equal to or less than zero, indicating that the entry probability is a decreasing function of debt ratio. Contrarily, it can be shown in figure 8 that the change rate of entry probability with respect to time index (time length) is equal to or greater than zero, indicating that the entry probability is an increasing function of time. Figures 7 and 8 confirm the sensitivity in figure 6. Figures 9-11 illustrate the impacts of price drift and volatility on the entry probability. At time 1, the entry probability could be a decreasing function of price volatility. However, as time elapses towards year 10, the entry probability gradually becomes an increasing function of price volatility as shown in figures 10 and 11. In contrast, the entry probability more consistently demonstrates an increasing function of price drift as shown in figures 9-11.

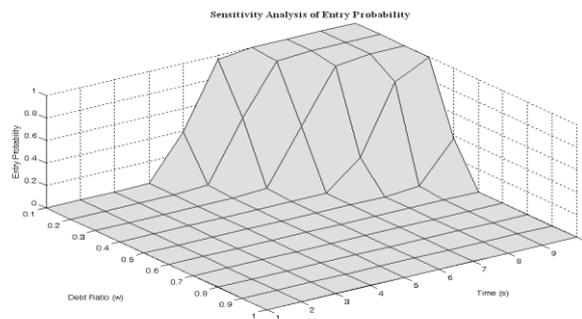


Figure 6: The sensitivity of entry probability to debt ratio and to time length (s) when t=0

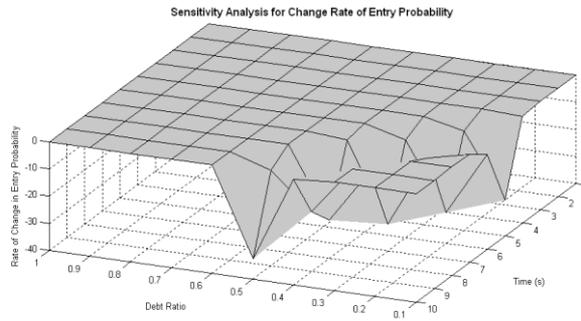


Figure 7: The sensitivity for rate of change in entry probability to debt ratio

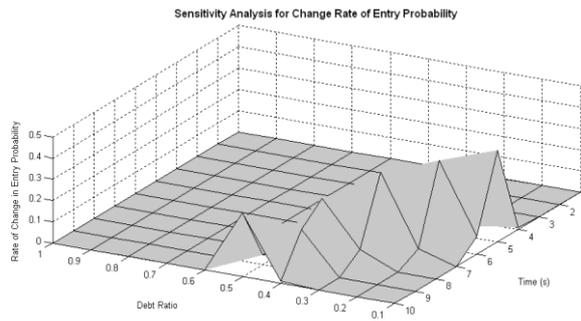


Figure 8: The sensitivity for rate of change in entry probability to time index (s)

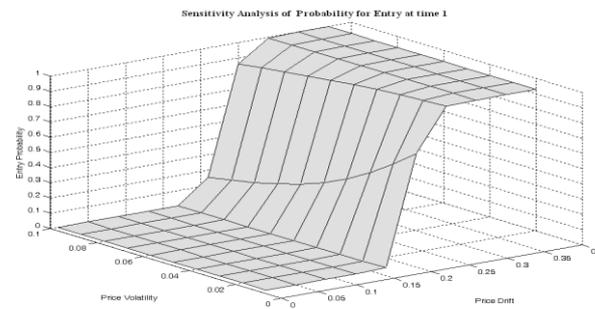


Figure 9: The sensitivity of entry probability to price drift and price volatility at time 1 (s=1)

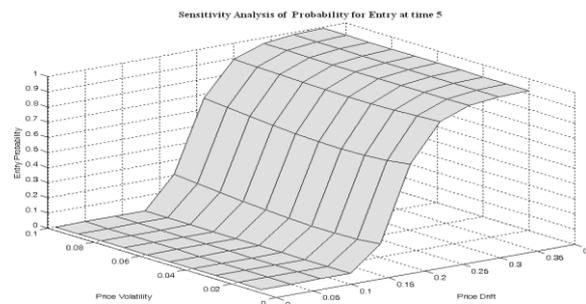


Figure 10: The sensitivity of entry probability to price drift and price volatility at time 5 (s=5)

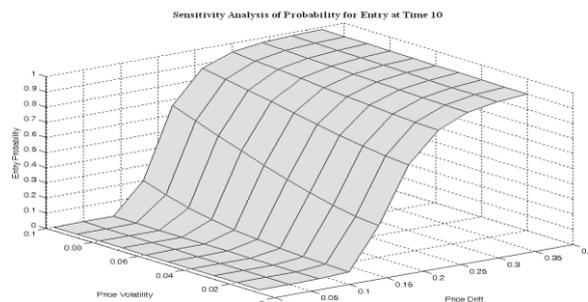


Figure 11: The sensitivity of entry probability to price drift and price volatility at time 10 ( $s=10$ )

## 4 Conclusion

This paper extends the work of Chen and Chang [14] by integrating the stochastic price in investment valuation to investigate the impacts of price uncertainty, tax, debt ratio, demand elasticity, and market share on investment threshold and on real option value. In this framework, the uncertainty of future profits comes from the stochastic price. To construct the value dynamics for the investment opportunity in question, I employ a stochastic value process driven by the stochastic price as Chen and Chang [14] proposed. The profit flows are discounted and summed up as expected investment project value. Analytical solutions for investment threshold, the real option value and the corresponding entry probability are obtained. Based on the analytical results, sensitivity of investment threshold, real option value, and entry probability are analyzed. The analytical result demonstrates that the higher the debt ratio, the higher the investment threshold is. In other words, it is less likely to undertake such an investment project when the debt ratio is higher. However, once the investment is undertaken, the higher the debt ratio, the higher the real option value is.

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## Appendix

Based on equation (25), the entry probability is as follows.

$$\begin{aligned} & \text{prob}(V_s \geq V^*) \\ &= 1 - N(k) \\ &= 1 - \int_{-\infty}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \end{aligned}$$

Taking the first derivative with respect to debt ratio  $w$ , we can derive the analytical solution as follows.

$$\begin{aligned} & \frac{\partial \text{prob}(V_s \geq V^*)}{\partial w} \\ &= - \frac{\partial \int_{-\infty}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}{\partial w} \\ &= - \frac{\partial \int_{-\infty}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}{\partial k} \cdot \frac{\partial k}{\partial w} \\ &= - \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{\partial k}{\partial w} \\ &= - \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{\partial \frac{\ln \frac{V^*}{V_t} - (u - 0.5v^2)(s-t)}{v\sqrt{s-t}}}{\partial w} \\ &= - \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{\partial \ln \frac{V^*}{V_t}}{\partial w} \\ &= - \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{\partial \ln \frac{V^*}{V_t}}{\partial w} \\ &= - \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{\partial \frac{V^*}{V_t}}{\partial w} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{c_1^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{\partial \left( \frac{\left(\frac{d_1}{d_1-1}\right) \cdot I}{(1-\gamma-wbr_D)(1-\eta)m P_t^{1-\varepsilon}} \right)}{\partial w} \\
 &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \\
 &\quad \times \frac{\partial \left( \frac{\left( \frac{\left(\frac{1}{2} - \frac{u}{v^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{u}{v^2}\right)^2 + \frac{2((r_D(1-\eta) - r_E)w + r_E)}{v^2}} \right)}{\left(\frac{1}{2} - \frac{u}{v^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{u}{v^2}\right)^2 + \frac{2((r_D(1-\eta) - r_E)w + r_E)}{v^2}} - 1 \right) \cdot I}{\frac{(1-\gamma-wbr_D)(1-\eta)m}{(r_D(1-\eta) - r_E)w + r_E - (\mu - \frac{1}{2}\sigma^2\varepsilon)(1-\varepsilon)} P_t^{1-\varepsilon}} \right)}{\partial w} \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{1}{m(1-\eta)P_t^{1-\varepsilon}} \cdot I \cdot \\
 &\quad \left( \frac{\left(\frac{d_1}{d_1-1}\right) \cdot \frac{(1-\gamma-wbr_D)(r_D(1-\eta) - r_E) + \left(\delta - (\mu - \frac{1}{2}\sigma^2\varepsilon)(1-\varepsilon)\right)(br_D)}{\left(\delta - (\mu - \frac{1}{2}\sigma^2\varepsilon)(1-\varepsilon)\right)^2} + \frac{(1-\gamma-wbr_D)}{\delta - (\mu - \frac{1}{2}\sigma^2\varepsilon)(1-\varepsilon)} \cdot \left(\frac{h^{-0.5} \cdot (r_D(1-\eta) - r_E)}{v^2 (d_1-1)^2}\right)}{\left(\frac{(1-\gamma-wbr_D)}{\delta - (\mu - \frac{1}{2}\sigma^2\varepsilon)(1-\varepsilon)}\right)^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{1}{m(1-\eta)P_t^{1-\varepsilon}} \cdot I \cdot \\
 &\quad \left( \frac{\left( \frac{d_1}{d_1-1} \right) \cdot \frac{(1-\gamma-wbr_D)(r_D(1-\eta)-r_E)}{\left( \delta - \left( \mu - \frac{1}{2} \sigma^2 \varepsilon \right) (1-\varepsilon) \right)} + (br_D) + \left( \frac{(1-\gamma-wbr_D)}{1} \right) \left( \frac{h^{-0.5} \cdot (r_D(1-\eta)-r_E)}{v^2 (d_1-1)^2} \right)}{\frac{(1-\gamma-wbr_D)^2}{\delta - \left( \mu - \frac{1}{2} \sigma^2 \varepsilon \right) (1-\varepsilon)}} \right) \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{1}{m(1-\eta)P_t^{1-\varepsilon}} \cdot I \cdot \\
 &\quad \left( \frac{\left( \frac{d_1}{d_1-1} \right) \cdot \frac{(1-\gamma-wbr_D)(r_D(1-\eta)-r_E)}{\left( \delta - \left( \mu - \frac{1}{2} \sigma^2 \varepsilon \right) (1-\varepsilon) \right)} + \frac{(br_D)}{\frac{(1-\gamma-wbr_D)^2}{\delta - \left( \mu - \frac{1}{2} \sigma^2 \varepsilon \right) (1-\varepsilon)}} - \frac{\left( \frac{(1-\gamma-wbr_D)}{1} \right) \left( \frac{h^{-0.5} \cdot (r_D(1-\eta)-r_E)}{v^2 (d_1-1)^2} \right)}{\frac{(1-\gamma-wbr_D)^2}{\delta - \left( \mu - \frac{1}{2} \sigma^2 \varepsilon \right) (1-\varepsilon)}} \right) \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{I}{m(1-\eta)P_t^{1-\varepsilon}} \\
 &\quad \cdot \left( \frac{(r_D(1-\eta)-r_E) \left( \left( \frac{d_1}{d_1-1} \right) + \left( \frac{h^{-0.5}}{v^2 (d_1-1)^2} \right) \left( \delta - \left( \mu - \frac{1}{2} \sigma^2 \varepsilon \right) (1-\varepsilon) \right) \right)}{(1-\gamma-wbr_D)} + \frac{(br_D) \left( \delta - \left( \mu - \frac{1}{2} \sigma^2 \varepsilon \right) (1-\varepsilon) \right)}{(1-\gamma-wbr_D)^2} \right) \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{2}} \cdot \frac{1}{v\sqrt{s-t}} \cdot \frac{V_t}{V^*} \cdot \frac{I}{m(1-\eta)P_t^{1-\varepsilon}} \cdot \left( \frac{\beta \left( \left( \frac{d_1}{d_1-1} \right) + \left( \frac{h^{-0.5}}{v^2 (d_1-1)^2} \right) \left( \delta - v \right) \right)}{\alpha} + \frac{(br_D)(\delta - v)}{\alpha^2} \right)
 \end{aligned}$$

where  $h = \left( \frac{1}{2} - \frac{u}{v^2} \right)^2 + \frac{2\delta}{v^2}$ ,

$$k = \frac{\ln \frac{V^*}{V_t} - (u - 0.5v^2)(s-t)}{v\sqrt{s-t}},$$

$$\alpha = 1 - \gamma - wbr_D,$$

and  $\beta = r_D(1-\eta) - r_E$ .