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On the Mathematical Definition of Cyberspace

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Abstract

We give a mathematical definition of cyberspace including basic specifications of its different formalities. To this end, we set an appropriate framework for determining adequate theoretical background, allowing rigorous, supple, univalent and adaptive description of what exactly we mean by saying "cyberspace".At the basis of this framework is the concept of the $e-$category $\boldsymbol{W_e}$. An$e-$categorycan be viewed as an infinite $e-graph(\mathbb{V},\mathbb{E})$withvector weights, in such a way that the $e-$nodes in $\mathbb{V}$ are the $e-$objects, while the$e-$edges or $e-$arcs in $\mathbb{E}$ are the $e-$morphisms. Given this notion, we investigate the possibility of allocating vector weights to objects and morphisms of any $e-$category $\boldsymbol{W_e}$. We also introduce a suitable metrizable topology on $e-$ graphs and $e-$ categories. The most significant benefits coming from the consideration of such a metric $\boldsymbol{d_{W_e}}$ in the set $\boldsymbol{ob}(\boldsymbol{W_e})$of objects of an $e-$categorycan be derived fromthe definitions of *cyber-evolution* and *cyber-domain*. Bearing all this in mind, we define the *local* $e-$ *dynamics*, as a mapping $\mathcal{cy}:[0,1]\rightarrow\left(|\boldsymbol{ob}(\boldsymbol{W_e})|,\boldsymbol{d_{W_e}}\right)$; its image is an$e-$*arrangement*. The points of an $e-$*arrangement* are the *instantaneous local* $e-$ *node manifestations*. An $e-$ arrangement together with all of its instant $e-$morphisms is an $e-$*regularization*. The elements of the completion $|\boldsymbol{ob}(\boldsymbol{W_e})|$ of the set $\boldsymbol{ob}(\boldsymbol{W_e})$ of objects of an $e-$categoryare the *cyber-elements*, while the topological space $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|,\boldsymbol{d_{W_e}}\right)$ is called a *cyber-domain*. A continuous local $e-$dynamics is said to be a *cyber-evolutionary path* or simply *cyber-evolution* of the

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cyber-domain. A cyber-arrangement together with its instantaneous homomorphisms is called a *cyberspace*. We investigate conditions under which an$e-$regularizationmay besusceptibleof aprojective$e-$limit. Subsequently, we define and discuss the concept of the *length* in a cyber-domain. The intrinsic cyber-metric is a metric possible to define on every cyber-domain. For this metric the distance between two cyber-elements is the length of the shortest cyber-track between these cyber-elements. We will conclude with a discussion about the *speed* of a cyber-evolution. Finally, we will give simple pointwise and uniform convergence of cyber-evolutions.

Keywords: theory of mathematical modeling, game-theoretic models, category of sets, internet topics,graph theory, applications of graph theory.

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1 Introduction

According to a commonly accepted formality given by the Oxford English Dictionary, 2009 Edition, **"cyberspace** is the notional environment in which communication over computer networks occurs" (www.oxforddictionaries.com/us/definition/american_english/cyberspace).However, during the entire period from 2009 until today, uses both the Internet and, in general, the networking and digital communications increased dramatically and now the term "cyberspace" is already able to represent many additional new ideas and phenomena.

Nevertheless, the concept of cyberspace suffers, always and clearly, the lack of a rigorous definition. Actually, there are many other formalities of the term in question. The most recent draft formality is the following: "cyberspace is a global and dynamic domain (subject to constant change) characterized by the combined use of electrons and electromagnetic spectrum, whose purpose is to create, store, modify, exchange, share and extract, use, eliminate information and disrupt physical resources (www.academia.edu/7096442/How_would_you_define_Cyberspace)."

The problem occurs particularly complex or even entangled, since "today, the word "cyberspace" is used in many contexts, but it is not always clear what exactly that term describes and what it means. Thus, different organizations have adopted different definitions of what cyberspace means. Some of them -- like the EU -- do not have an official definition at all, but that does not prevent it from discussing the term (http://blogs.cisco.com/security/cyberspace-what-is-it)".

Here is an attempt to produce indicative ontology of cyberspace by using other various formalities created by multiple national government and relevant international bodies.

1st Definition: United Kingdom, the UK Cyber Security Strategy, 2011

- "Cyberspace is an interactive domain made up of digital networks that is used to store, modify and communicate information. It includes the internet, but also the other information systems that support our businesses, infrastructure and services."

2nd Definition: United States, National Security Presidential Directive 54/Homeland Security Presidential Directive 23, 2008

- "Cyberspace is defined as the interdependent network of information technology infrastructures, and includes the Internet, telecommunications networks, computer systems, and embedded processors and controllers in critical industries. Common usage of the term also refers to the virtual environment of information and interactions between people."

3rd Definition: European Union, Glossary | Europe -- Information Society (Archived), unknown year

- "Word invented by the writer William Gibson in his play "le Neuromacien". It describes the virtual space in which the electronic data of worldwide PCs circulate."

4th Definition: Canada, Canada's Cyber Security Strategy, 2010

- "Cyberspace is the electronic world created by interconnected networks of information technology and the information on those networks. It is a global commons where more than 1.7 billion people are linked together to exchange ideas, services and friendship."

5th Definition: New Zealand, New Zealand Cyber Security Strategy, 2011

- "The global network of interdependent information technology infrastructures, telecommunications networks and computer processing systems in which online communication takes place."

6th Definition: Germany, Cyber Security Strategy for Germany, 2011

- "Cyberspace is the virtual space of all IT systems linked at data level on a global scale. The basis for cyberspace is the Internet as a universal and publicly accessible connection and transport network which can be complemented and further expanded by any number of additional data networks. IT systems in an isolated virtual space are not part of cyberspace."

7th Definition: ISO/IEC, ISO/IEC 27032 Guidelines for cybersecurity (DRAFT), 2011

- "The complex environment resulting from the interaction of people, software and services on the Internet by means of technology devices and networks connected to it, which does not exist in any physical form."

8th Definition: ITU, ITU-T Recommandation Rec. ITU-T X.1205 (X.cso), 2008

- "Technologies, such as wireless networks and voice-over-IP (VoIP), extend the reach and scale of the Internet. In this regard, the cyber environment includes users, the Internet, the computing devices that are connected to it and all applications, services and systems that can be connected directly or indirectly to the Internet, and to the next generation network (NGN) environment, the latter with public and private incarnations. Thus, with VoIP technology, a desk telephone is part of the cyber environment. However, even isolated devices can also be part of cyber environment if they can share information with connected computing devices through removable media. The cyber environment includes the software that runs on computing devices, the stored (also transmitted) information on these devices or information that are generated by these devices. Installations and buildings that house the devices are also part of the cyber environment."

Having regard to the above variety of definitions, there have been several official but non rigorous descriptions for the related concept of cyber-security. For instance, we give formal views for cyber-security, as adopted from Australia and The Netherlands.

9th Definition: Australia, Cyber Security Strategy -- An Overview, 2009

- "Cyber security refers to the safety of computer systems – also known as information and communications technologies (or ICT)."

10th Definition: The Netherlands, the National Cyber Security Strategy, 2011

- "Cyber security is to be free from danger or damage caused by disruption or fall-out of ICT or abuse of ICT. The danger or the damage due to abuse, disruption or fall-out can be comprised of a limitation of the availability and reliability of the ICT, breach of the confidentiality of information stored in ICT or damage to the integrity of that information."

All these formalities are abstract and introduce the interested reader to the spirit of the text, but, on the other hand, they may not give a univocal, literal and rigorous description of the concept. On the contrary, the availability of more than 27 various such formalities can be confusing... Furthermore, there is no relevant mathematical background guarantying coherent development and systemic safety, unlike other scientific fields which have been built on solid mathematical foundations. Moreover, in all these formulations, there is nowhere any provision for the perpetual change of cyberspace. Indeed, it is clear that, each time moment, appear new nodes that connect to the cyberspace in different points of the Earth (even of the Space), while at the same time moment, other nodes cease to be connected, so the points displaying their presence in cyberspace disappear completely. On the other hand, there are nodes that are connected but do not have an active connection with the cyberspace. In addition, there is no available strict description of the activities and interactions between parts of cyberspace.

Inevitably, the lack of a rigorous definition has the consequence that they have developed and are continually being developed varying techniques, without any mathematical foundation which can support the best and efficient solutions on the issues of a beneficial settlement and utilization of these techniques. So, it is no longer possible to organize a soliddefense (or planned attack) in the cyberspace, if previously there has not been a closed and rigorous mathematical definition of cyberspace. Nor is it possible to guarantee effective protection into the cyberspace, if previously there has not been a strict definition of the concept of cyber-attack, as well as the different types of these attacks.

Until now, only a few isolated efforts have been made to provide a single theory for the modeling of cyberspace (see, for instance, [1], [24] and [29]).The overwhelming majority of scientific work that has been done so far concerns cyber-security modeling issues (see, for instance, [4-6], [21-23], [26-28] and [31]).The main aim of this paper is to give a rigorous mathematical definition of cyberspace, which will include all the basic specifications of the above mentioned different formalities of the cyberspace, as well as the remarks followed these formalities.To do so, in the first three Sections, we will set the appropriate theoretical framework for determining adequate mathematical background, allowing the rigorous, supple, univalent, and adaptive description of what exactly we mean by saying "cyberspace". Specifically,in Section 1, we will consider the (infinite) set $\boldsymbol{\mathcal{M}nf_e}$ of all pairs$(\boldsymbol{V}, \boldsymbol{E})$of the so-called $e-$ *node manifestations* $\boldsymbol{V}$ and interconnected $e-$ *edge manifestations* $\boldsymbol{E}$ of instantaneous virtual $e-$*archetype germs* (with $N-$layers) over a given geographical region $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$. A good thinking for introducing utilitarian algebraic structure in the set $\boldsymbol{\mathcal{M}nf_e}$is to see this set as the class of objects of a large category. Based on this preliminary and preparatory material, in Section 2, we will introduce the $e-$*graph category*$\boldsymbol{\mathcal{E}_{\mathcal{C}}}$, whose objects $\boldsymbol{X} \in \boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$ are simply the pairs $\boldsymbol{X} = (\boldsymbol{V}, \boldsymbol{E}) \in \boldsymbol{\mathcal{M}nf_e}$. For later use, we will also introduce four other $e-$ categories: the $e-$ *set category* $\boldsymbol{e_{Set}}$ whose objects are the subsets of $\boldsymbol{\mathcal{E}_{\mathcal{C}}}$, the $e-$ *homomorphism category*whose objects are sets of homomorphisms between subsets of $\boldsymbol{e_{Set}}$, the $e-$*group category* whose objects are the groups of $\boldsymbol{\mathcal{E}_{\mathcal{C}}}$ and the $e-$*topological category*$\boldsymbol{e_{Top}}$whose objects are topological subcategories of $\boldsymbol{\mathcal{E}_{\mathcal{C}}}$. The Section will conclude with a brief initial study of functors between these $e-$categories. Having regard to the above, in the next Section 3,after foundation of the so-called $e-$*universality* in a categorical construction, we will define the $e-$*product* and the $e-$*coproduct* between any two objects lying in an $e-$category of the above type. With this background, we will be able to proceed to the main goal of the paper. In this direction, Section 4 will investigate the possibility of allocating suitable vector weights to all the objects and morphisms of any $e-$ category $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e} = \{\boldsymbol{\mathcal{E}_{\mathcal{C}}}, \boldsymbol{e_{Set}}, \boldsymbol{e_{Gpr}}, \boldsymbol{e_{Top}}\}$. Towards this end, we will consider two types of vector weights that can be attached to any object and/or morphism of such an $e-$category: the maximum weight and the square weight. Any such weight will be a point in the positive quadrant of the plane. Taking this into account, in Section 5, any $e-$ category $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e} = \{\boldsymbol{\mathcal{E}_{\mathcal{C}}}, \boldsymbol{e_{Set}}, \boldsymbol{e_{Hom}}, \boldsymbol{e_{Gpr}}, \boldsymbol{e_{Top}}\}$ will be viewed as an infinite $e-$*graph*$(\mathbb{V}, \mathbb{E})$ with vector weights, in such a way that the $e-$nodes in $\mathbb{V}$ are the $e-$ objects $\boldsymbol{X} \in \boldsymbol{ob}(\boldsymbol{W_e})$, while the $e-$ edges or $e-$ arcs in $\mathbb{E}$ are the $e-$morphisms $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{W_e})$. For such an $e-$graph $\mathfrak{G}_{\boldsymbol{W_e}}$ corresponding to an $e-$ category $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e} = \{\boldsymbol{\mathcal{E}_{\mathcal{C}}}, \boldsymbol{e_{Set}}, \boldsymbol{e_{Hom}}, \boldsymbol{e_{Gpr}}, \boldsymbol{e_{Top}}\}$, the vector weight of the $e-$node associated to the $e-$manifestation $\boldsymbol{X} = (\boldsymbol{V_X}, \boldsymbol{E_X}) \in \mathbb{V} \equiv \boldsymbol{ob}(\boldsymbol{W_e})$ is equal to a weight of $\boldsymbol{X}$. Bearing all this in mind, in Section 6, we will introduce a suitable metrizable topology on $e-$ graphs and $e-$ categories. The most significant benefits coming from the consideration of such an appropriate metric $\boldsymbol{d_{W_e}}$ in the set $\boldsymbol{ob}(\boldsymbol{W_e})$ of objects of an $e-$category $\boldsymbol{W_e}$ can be derived from the definitions of *cyber-evolution* and *cyber-domain*. To do this, in Section 7, we will first define the concept of *local* e*—dynamics*, as a mapping of the form $\mathcal{cy}: [0,1] \to (\boldsymbol{ob}(\boldsymbol{W_e}), \boldsymbol{d_{W_e}})$; its image $\mathcal{cy}(\mathbb{I})$ is an e*—arrangement*. Each point $\mathcal{cy}(\boldsymbol{t}) \in \mathcal{cy}(\mathbb{I})$ is an instantaneous local $e-$node manifestation with an interrelated $e-$edge manifestation. An $e-$arrangement together with all of its instant e—morphisms is an $e-$*regularization*. If the given geographical region $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$ is selected to be a cover of the whole domain of interest, then the above mapping is called a *global* $e-$*dynamics*. In such a case, the image $\mathcal{cy}(\mathbb{I})$ is a global $e-$*arrangement* and the associated $e-$*regularization* is a $e-$*settlement*. The elements of the completion $|\boldsymbol{ob}(\boldsymbol{W_e})|$ of $\boldsymbol{ob}(\boldsymbol{W_e})$ in $\overline{\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}} \subset \mathbb{C}\mathbf{P}^{\boldsymbol{N}}$ are the *cyber-elements* while the topological space $(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}})$ is a *cyber-domain*. With this notation, a continuous local $e-$dynamics $\mathcal{cy}: [0,1] \to (|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}})$ is said to be a *cyber-evolutionary path* or simply *cyber-evolution* of the cyber-domain $(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}})$. Sometimes, we may also use the compellation *cyber-track* of $(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}})$. In such a case, the e—arrangement $\mathcal{cy}(\mathbb{I})$ is said to be a *cyber-arrangement*. A cyber-arrangement together with all of its instantaneous homomorphisms is called a *cyberspace*. If, in particular, the given geographical region $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$ is selected to be a cover of the whole domain of interest, then the corresponding cyber-arrangement $\mathcal{cy}(\mathbb{I})$ is said to be a *cyber-configuration*; the associated cyberspace is also called *cyberspace*. In view of the above concepts, Section 8 will investigate conditions under which an $e-$ regularizationmay besusceptibleof aprojective$e$ $-$limit. As we shall see, it is important to know if a sub-$e-$ regularization is projective e $-$system. Subsequently, in Section 9 we will define and discuss the concept of the *length* in a cyber-domain. The intrinsic cyber-metric is a metric possible to define on every cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d}_{\boldsymbol{W_e}}\right)$. For this metric the distance between two cyber-elements is the lengthof the "shortest cyber-track" between these cyber-elements. The term shortest cyber-track will be defined later and is in fact crucial for the understanding of cyber-geodesics. Although every shortest cyber track on a cyber-length space is a cyber-geodesic, the reverse argument is not valid. In fact, some cyber-geodesics may fail to be shortest cyber-tracks on large scales. All the same, since each cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d}_{\boldsymbol{W_e}}\right)$ is a compact, complete metric space, and since for any pair of cyber-elements in $|\boldsymbol{ob}(\boldsymbol{W_e})|$ there is a cyber-evolutionary path of finite length joining them, one obtains the following converse result: any pair of two cyber-elements in each cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d}_{\boldsymbol{W_e}}\right)$ has a shortest cyber track joining them. Section 9 will conclude with a discussion about the *speed* (: *cyber-speed*) of a cyber-evolution. Finally, in the last Section 10, we will give simple pointwise and uniform convergence of cyber-evolutions.

2 Mathematical definition of cyberspace

When it is pronounced the word "cyberspace" in our mind spontaneously comes the meaning of a network. However, this may be understood as applying only for infinitesimal time intervals. Forwarded in support of this assertion, we recall that anyone can be connected to or disconnected from the so called "cyberspace", for example, using a mobile phone, from any point on the earth's surface and at any time he wishes. This means that the quasi "network" that we believe represents the so-called "cyberspace" may acquire at any time, new nodes and new edges (: arcs -flows), but also it may lose existing nodes or edges (: arcs -flows), so it is necessary to disprove any current perception about a stable structure of cyberspace that remains constant throughout the year. The issue becomes more essential if we realize that a modern and central security goal presupposes ensuring an accurate predictive model of possible temporal and geographical points where there will be a serious possibility for the manifestation of an attack against targets in the "cyberspace". It follows that the concept of the so called' cyberspace" is governed by two main features: **it has variable character in the space-time and unstable nature**.

Moreover, it is established and accepted that the so-called "cyberspace" manifests itself through at least 5 (five) interconnected layers ([18]):

- *the physical web layer* (composed of the physical devices that contribute to the construction of cyberspace),
- *the systemic web layer* (composed of the logical decisions that shape the so-called "cyberspace "),
- *the web layer of applications* (which consists of applications entering and moving into the so-called "cyberspace" through exchange of information),
- *the human layer* (comprising the human plurality which enters or exits the so-called "cyberspace") and
- *the layer of process*.

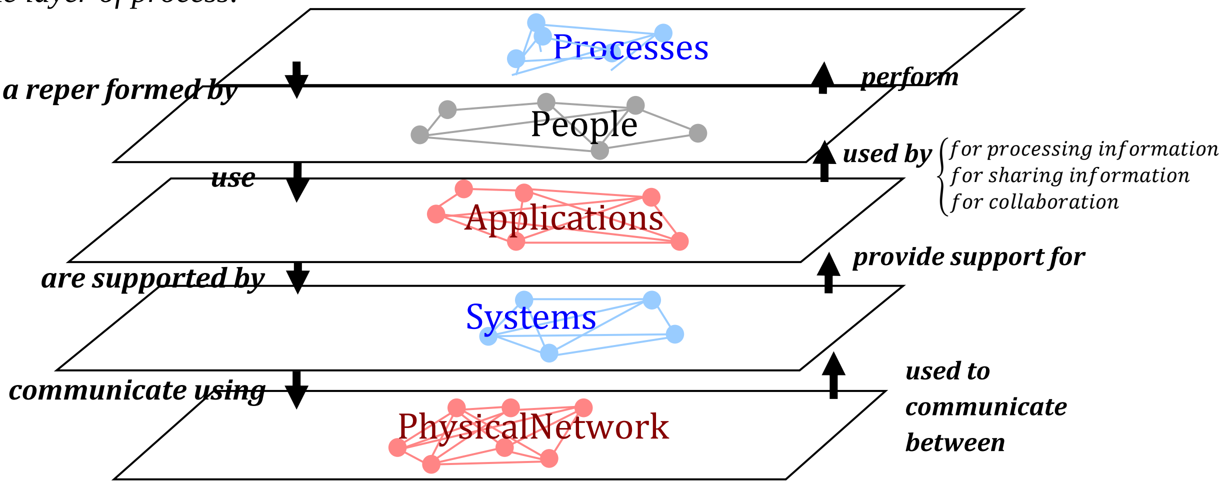

Figure 1: Interlayer relationships of cyberspace ([30])

For reasons of possible future extension, we will prefer to consider the general case of a weighted network (or graph) $\boldsymbol{\mathcal{X}}$ with several interconnected layers (rather than the reported five).

Definition 2.1 The multilayered weighted (finite or infinite) graph $\boldsymbol{\mathcal{X}}$ with $\boldsymbol{N}$ interconnected layers is said to bean $\boldsymbol{N}-$ **cyber-archetype germ** or an **instantaneous virtuale** $-$**archetype germ with** $\boldsymbol{N}$ $-$**layers.**■

Embedding an $\boldsymbol{N}-$cyber-archetype germ $\boldsymbol{\mathcal{X}}$ into the Cartesian product $\boldsymbol{\Sigma_1} \times \ldots \times \boldsymbol{\Sigma_N}$ of $\boldsymbol{N}$ Euclidean sets $\boldsymbol{\Sigma_1} \subset \mathbb{R}^{\boldsymbol{n_1}}, \ldots, \boldsymbol{\Sigma_N} \subset \mathbb{R}^{\boldsymbol{n_N}}$ (or, more generally, of $\boldsymbol{N}$ subsets $\boldsymbol{\Sigma_1} \subset \mathbb{C}\mathbf{P}^{\boldsymbol{n_1}}, \ldots, \boldsymbol{\Sigma_N} \subset \mathbb{C}\mathbf{P}^{\boldsymbol{n_N}}$ of complex projective space $\mathbb{C}\mathbf{P}^{\boldsymbol{n_k}} \equiv \mathbf{P}(\mathbb{C}^{\boldsymbol{n_k+1}})$) gives a geographical qualifier at each node of $\boldsymbol{\mathcal{X}}$.

Definition 2.2 An **instantaneous $\boldsymbol{e}$ −manifestation**, or simply **$\boldsymbol{e}$ −manifestation**, of the $\boldsymbol{N}-$cyber archetype germ $\boldsymbol{\mathcal{X}}$ in $\boldsymbol{\Sigma_1}, \ldots, \boldsymbol{\Sigma_N}$ is a representation $\boldsymbol{\mathcal{X}} \to \boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$ of $\boldsymbol{\mathcal{X}}$ into a Cartesian product $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N} \subseteq \boldsymbol{\Sigma_1} \times \ldots \times \boldsymbol{\Sigma_N}$ such that

- all nodes of $\boldsymbol{\mathcal{X}}$ in the $\boldsymbol{\alpha}-$layer, called **instantaneous $\boldsymbol{e}$ −node manifestations**, or simply **$\boldsymbol{e}$ −node manifestations**, are illustrated at weighted points of the set $\boldsymbol{U_\alpha}$
- all directed edges (flows) of $\boldsymbol{\mathcal{X}}$ in the $\boldsymbol{\alpha}-$layer, called **instantaneous $\boldsymbol{e}$ −edge manifestations**, or simply **$\boldsymbol{e}$ −edge manifestations**, are given by simple weighted edges, i.e. by weighted homeomorphic images of the closed interval $[\mathbf{0}, \mathbf{1}]$ on the set $\boldsymbol{U_\alpha}$, so that
- for any $\boldsymbol{\alpha} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{N}$, the end points of each $e-$edge manifestation on the set $\boldsymbol{U_\alpha}$ must be images of end points of a corresponding original directed edge of $\boldsymbol{\mathcal{X}}$ in the $\boldsymbol{\alpha}-$layer
- for any $\boldsymbol{\alpha} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{N}$, there should not be any $e-$edge manifestation on the set $\boldsymbol{U_\alpha}$ derived from directed $e-$edge of $\boldsymbol{\mathcal{X}}$ in the $\boldsymbol{\alpha}-$layer into which belong points of $e-$edge manifestations that are defined by other nodes of $\boldsymbol{\mathcal{X}}$ in the same layer (or, equivalently, two $e-$edge manifestations in the set $\boldsymbol{U_\alpha}$ originating from different directed edges of $\boldsymbol{\mathcal{X}}$ in the $\boldsymbol{\alpha}-$layer do not intersect at any interior point of these edges). ■

Remark 2.1 Each instantaneous local manifestation of cyber-archetype germ is an embedding of a weighted graph in $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$ that attributes a geographical qualifier at each node in the $\boldsymbol{\alpha}-$ layer of the relative $\boldsymbol{N}-$cyber-archetype germ. For example, any weighted node of the physical network layer holds a weighted geographical position defined by an instantaneous local $e-$manifestation. Similarly, any weighted node on the layer with human users also occupies a weighted geographical position which again is determined by the instantaneous local $e-$manifestation of the relevant network on another copy of the geographic environment. ■

Remark 2.2 The introduction of subsets $\boldsymbol{U_1}, \dots, \boldsymbol{U_N}$ gives us the option of investigating instantaneous **local**e $-$manifestations of an $\boldsymbol{N}$ $-$cyber-archetype germ. If one wishes to conduct a full global study, it is sufficient to take $\boldsymbol{U_1} = \boldsymbol{S_1}$, ..., $\boldsymbol{U_N} = \boldsymbol{S_N}$. A key generic question that is raised reasonably is as follows. **"**Is it possible to carry out a complete study of instantaneous e $-$manifestations over the Cartesian product $\boldsymbol{\Sigma_1} \times \dots \times \boldsymbol{\Sigma_N}$ of entire regions $\boldsymbol{\Sigma_1}, \dots, \boldsymbol{\Sigma_N}$ from a finite (or discrete) number of studies of instantaneous e $-$manifestations conducted over the Cartesian product $\boldsymbol{U_1^{(k)}} \times \dots \times \boldsymbol{U_N^{(k)}}$ of local (small) regions $\boldsymbol{U_1^{(k)}}, \dots, \boldsymbol{U_N^{(k)}}$ ($\boldsymbol{k = 1, 2, \dots, M}$ for some $\boldsymbol{M} \in \mathbb{N}_{\boldsymbol{0}} \cup \{\infty\}$), such that $\boldsymbol{\Sigma_1} \times \dots \times \boldsymbol{\Sigma_N} = \bigcup_{\boldsymbol{k=1}}^{\boldsymbol{M}} \left(\boldsymbol{U_1^{(k)}} \times \dots \times \boldsymbol{U_N^{(k)}} \right)$? "This question escapes the scope of this study and could be the subject of further research. ■

It is obvious that *the potential* e*— node manifestations* $\boldsymbol{V}$ *of an* $\boldsymbol{N}$ $-$ *cyber archetype germ* $\boldsymbol{\mathcal{X}}$ *over a Cartesian area* $\boldsymbol{U_1} \times \dots \times \boldsymbol{U_N}$ *are infinite*. It is also clear that, *for such each* e*—node manifestation* $\boldsymbol{V}$ *of* $\boldsymbol{\mathcal{X}}$, *there is an interrelated finite set* $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{V})$ *of possible Cartesian products of weighted edges representing* e*—edge manifestations* (:"painless flows" or "malicious flows").

Definition 2.3 The (infinite) set $\boldsymbol{\mathcal{M}nf_e} = \boldsymbol{\mathcal{M}nf_e}(\boldsymbol{U_1} \times \dots \times \boldsymbol{U_N})$ of all such pairs $(\boldsymbol{V}, \boldsymbol{E})$ of e $-$node manifestations $\boldsymbol{V}$ and interconnected e $-$edge manifestations $\boldsymbol{E}$ of $\boldsymbol{N}$ $-$cyber archetype germs in $\boldsymbol{U_1} \times \dots \times \boldsymbol{U_N}$ will be called the **superclass of cyber-manifestations,** or simply the $\boldsymbol{e}$ **$-$superclass,** in $\boldsymbol{U_1} \times \dots \times \boldsymbol{U_N}$.■

For obvious reasons and in order to facilitate the achievement of results, the$e$ $-$superclass $\boldsymbol{\mathcal{M}nf_e}$will beendowed withappropriatemathematicalstructures. To this end, the first to be appointed is an easy **algebraic** structure in the(infinite) set of all these e— manifestations $(\boldsymbol{V}, \boldsymbol{E})$ and simultaneously, a compatible **topological** structure to allow for a detailed analytic study of $\boldsymbol{\mathcal{M}nf_e}$.

3 e $-$ categories and e $-$functors

A good thinking for introducing utilitarian algebraic structure in the set $\boldsymbol{\mathcal{M}nf_e}$is to see this set as the class of objects of a large category.

Definition 3.1 The $\boldsymbol{e}$**—graph category** (or **web graph category**) $\boldsymbol{\mathcal{E}_{\mathcal{C}}} = \boldsymbol{\mathcal{E}_{\mathcal{C}}}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N})$ is a category consisting of the following three mathematical entities.

- **i.** The class $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$, whose elements are called $\boldsymbol{e}$**—objects** contains the pairs $\boldsymbol{X} = (\boldsymbol{V}, \boldsymbol{E}) \in \boldsymbol{\mathcal{M}nf_e}$ of instantaneous $e-$ node manifestations $\boldsymbol{V}$ and instantaneous interrelated $e-$edge manifestations $\boldsymbol{E}$ of $\boldsymbol{N}-$cyber archetype germs in the Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$.
- **ii.** The class $\boldsymbol{hom}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$, whose elements are called $\boldsymbol{e}$**—morphisms** on $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$. Each morphism $\boldsymbol{h}$ has a **source** e**—object** $\boldsymbol{X}$ and a **target** e**—object** $\boldsymbol{Y}$, in such a way that ***each*** $\boldsymbol{e}$ ***−object is the source of an*** $\boldsymbol{e}$ ***−morphism and the target of another***. The expression $\boldsymbol{h}: \boldsymbol{X} \longmapsto \boldsymbol{Y}$, would be verbally stated as "$\boldsymbol{h}$ is an $e-$ morphism from the manifestation $\boldsymbol{X}$ to the manifestation $\boldsymbol{Y}$". The expression $\boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Y})$ denotes the **hom-class** of all $e-$morphisms from $\boldsymbol{X}$ to $\boldsymbol{Y}$.
- **iii.** The binary operation $\circ$, called e**—composition of** e**—morphisms**, such that for any three $e-$objects $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$, we have $\boldsymbol{hom}(\boldsymbol{Y}, \boldsymbol{Z}) \times \boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Y}) \rightarrow \boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Z})$. The e—composition of $\boldsymbol{h}: \boldsymbol{X} \longmapsto \boldsymbol{Y}$ and $\boldsymbol{g}: \boldsymbol{Y} \longmapsto \boldsymbol{Z}$ is written as $\boldsymbol{g} \circ \boldsymbol{h}$ or $\boldsymbol{gh}$, governed by two axioms:
    - $\boldsymbol{e}$**— Associativity**: If $\boldsymbol{f}: \boldsymbol{X} \longmapsto \boldsymbol{Y}$, $\boldsymbol{g}: \boldsymbol{Y} \longmapsto \boldsymbol{Z}$, $\boldsymbol{h}: \boldsymbol{Z} \longmapsto \boldsymbol{T}$, then $\boldsymbol{h} \circ (\boldsymbol{g} \circ \boldsymbol{f}) = (\boldsymbol{h} \circ \boldsymbol{g}) \circ \boldsymbol{f}$, and
    - $\boldsymbol{e}$**— Identity**: For every $e-$object $\boldsymbol{X}$, there exists an $e-$morphism $\boldsymbol{1_X}: \boldsymbol{X} \longmapsto \boldsymbol{X}$ called the $\boldsymbol{e}$ **−identity morphism** for $\boldsymbol{X}$, such that for every $e-$morphism $\boldsymbol{f}: \boldsymbol{X} \longmapsto \boldsymbol{Y}$, we have $\boldsymbol{1_y} \circ \boldsymbol{f} = \boldsymbol{f} = \boldsymbol{f} \circ \boldsymbol{1_X}$. ■

Remark 3.1 From these axioms, it can be proved that there exists only one e—identity morphism for every $e-$object. ■

Generalizing, one may consider additionally the following other four basic $\boldsymbol{e}$**—categories**.

1. The $\boldsymbol{e}$ **−set category** $\boldsymbol{e_{Set}} = \boldsymbol{e_{Set}}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N})$ where the objects are all small subsets of $\boldsymbol{\mathcal{E}_{\mathcal{C}}} = \boldsymbol{\mathcal{E}_{\mathcal{C}}}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N})$.
2. The $\boldsymbol{e}-$ **homomorphism category** $\boldsymbol{e_{Hom}} = \boldsymbol{e_{Hom}}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N})$ where the objects are sets of homomorphisms between subsets of $\boldsymbol{e_{Set}}$.

3. The $\boldsymbol{e}-$**group category** $\boldsymbol{e_{Gpr}} = \boldsymbol{e_{Gpr}}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N})$ where the objects are the groups of $\boldsymbol{\mathcal{E}_C}$.
4. The $\boldsymbol{e}-$**topological category** $\boldsymbol{e_{Top}} = \boldsymbol{e_{Top}}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N})$ where the objects are topological subcategories of $\boldsymbol{\mathcal{E}_C}$.

For reasons of homogenization of symbolism, we will adopt the following common notation

$$\boldsymbol{\mathcal{W}_e} = \{\boldsymbol{\mathcal{E}_C}, \boldsymbol{e_{Set}}, \boldsymbol{e_{Hom}}, \boldsymbol{e_{Gpr}}, \boldsymbol{e_{Top}}\}.$$

The objects of each $e-$category $\boldsymbol{W_e} = \boldsymbol{W_e}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}) \in \boldsymbol{\mathcal{W}_e}$ will be called **instantaneous local $\boldsymbol{e}$-manifestations** or simply again $\boldsymbol{e}$ **−manifestations**.

For illustrative purposes, let me give you the detailed presentation of the overall form of an object in each of the above categories.

- If $\boldsymbol{W_e} = \boldsymbol{\mathcal{E}_C}$, then an $e-$ manifestation $\boldsymbol{X} \in \boldsymbol{\mathcal{E}_C}$ is a pair $\boldsymbol{X} = (\boldsymbol{V}, \boldsymbol{E})$ of instantaneous local manifestations of e −nodesand instantaneous interrelated local manifestations of e −edges on a$\boldsymbol{N}-$ cyber-archetype germ in the Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$, together with its instantaneouslocalmanifestationsof e −morphisms $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Y})$ with $\boldsymbol{Y} \in \boldsymbol{\mathcal{E}_C}$.
- If $\boldsymbol{W_e} = \boldsymbol{e_{Set}}$, then an e −manifestation $\boldsymbol{X} = \boldsymbol{K} \in \boldsymbol{e_{Set}}$ is a whole set of pairs $(\boldsymbol{V}, \boldsymbol{E})$ of instantaneous local manifestations of $e-$ nodes and instantaneous interrelated local manifestations of e −edges on a$\boldsymbol{N}-$ cyber-archetype germ in the Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$, together with its instantaneouslocalmanifestationsof e −morphisms$\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{K}, \boldsymbol{M})$ with $\boldsymbol{M} \in \boldsymbol{e_{Set}}$.
- If $\boldsymbol{W_e} = \boldsymbol{e_{Hom}}$, then an e −manifestation $\boldsymbol{X} = \mathbb{h} \in \boldsymbol{e_{Hom}}$ is a whole set $\{\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{K}, \boldsymbol{M})\}$ of e −morphisms in $\boldsymbol{e_{Set}}$, together with its e −morphisms $\boldsymbol{h} \in \boldsymbol{hom}(\mathbb{h}, \mathbb{g})$ with $\mathbb{g} \in \boldsymbol{e_{Hom}}$.
- If $\boldsymbol{W_e} = \boldsymbol{e_{Gpr}}$, then an $e-$manifestation $\boldsymbol{X} = \boldsymbol{G} \in \boldsymbol{e_{Gpr}}$ is a group of pairs $(\boldsymbol{V}, \boldsymbol{E})$ of instantaneous local manifestations of $e-$ nodesand instantaneous interrelated local manifestations of e −edges on a$\boldsymbol{N}-$ cyber-archetype germ in the Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$, together with its instantaneous local manifestations of e −morphisms $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{G}, \boldsymbol{H})$ with $\boldsymbol{H} \in \boldsymbol{e_{Gpr}}$.
- And, finally, if $\boldsymbol{W_e} = \boldsymbol{e_{Top}}$, then an $e-$ manifestation $\boldsymbol{X} = \boldsymbol{\Omega} \in \boldsymbol{e_{Top}}$ is a (Grothendieck or other) topological subcategory of $\boldsymbol{\mathcal{E}_C}$ consisting of pairs $(\boldsymbol{V}, \boldsymbol{E})$of instantaneous local manifestations of $e-$ nodes and instantaneous interrelated

local manifestations of e −edges on a$\boldsymbol{N}-$ cyber-archetype germ in the Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$,together with its instantaneous local manifestations of e −morphisms $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{\Omega}, \mathbf{D})$ with $\boldsymbol{D} \in \boldsymbol{e_{Top}}$.

Definition 3.2 Let $\boldsymbol{W_e}$ be any $e-$ category and let $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{W_e}$. An $e-$morphism $\boldsymbol{f} \in \boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Y})$ is an $\boldsymbol{e}$ **−isomorphism** between the two e −manifestations $\boldsymbol{X}$ and $\boldsymbol{Y}$ if there is an e −morphism $\boldsymbol{g} \in \boldsymbol{hom}(\boldsymbol{Y}, \boldsymbol{X})$ with $\boldsymbol{g} \circ \boldsymbol{f} = \boldsymbol{id_X}$ and $\boldsymbol{f} \circ \boldsymbol{g} = \boldsymbol{id_Y}$.■

Let $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e}$ and $\boldsymbol{W'_e} \in \boldsymbol{\mathcal{W}_e}$ be two e −categories. (It is not excluded to be equal). An **covariant $\boldsymbol{e}$ −functor** (respectively, **contravariant $\boldsymbol{e}$ −functor**) is a map $\boldsymbol{\mathcal{T}}: \boldsymbol{W_e} \to \boldsymbol{W'_e}$ assigning

- to each e −manifestation $\boldsymbol{X} \in \boldsymbol{W_e}$, an e −manifestation $\boldsymbol{Y} \in \boldsymbol{W'_e}$ and
- to each $e-$ morphism $\boldsymbol{f} \in \boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Y})$, an $e-$ morphism $\boldsymbol{\mathcal{T}}(\boldsymbol{f}) \in \boldsymbol{hom}\big(\boldsymbol{\mathcal{T}}(\boldsymbol{X}), \boldsymbol{\mathcal{T}}(\boldsymbol{Y})\big)$such that

$$\boldsymbol{\mathcal{T}}(\boldsymbol{id_X}) = \boldsymbol{id_{T(X)}} \text{ and } \boldsymbol{\mathcal{T}}(\boldsymbol{g} \circ \boldsymbol{f}) = \boldsymbol{\mathcal{T}}(\boldsymbol{f}) \circ \boldsymbol{\mathcal{T}}(\boldsymbol{g})$$

(respectively,

$$\boldsymbol{\mathcal{T}}(\boldsymbol{g} \circ \boldsymbol{f}) = \boldsymbol{T}(\boldsymbol{g}) \circ \boldsymbol{T}(\boldsymbol{f})).■$$

Definition 3.3 A **natural $\boldsymbol{e}-$ transformation** $\mathfrak{A}_{W_e}: \boldsymbol{\mathcal{T}} \longmapsto \boldsymbol{\mathcal{S}}$ also called **$\boldsymbol{e}$ −morphism of $\boldsymbol{e}$ −functors**, between e −functors $\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{S}}: \boldsymbol{W_e} \to \boldsymbol{W'_e}$ associates to each e −manifestation$\boldsymbol{X} \in \boldsymbol{W_e}$an e −morphism $\mathfrak{A}_{\boldsymbol{X}}$: $\boldsymbol{\mathcal{T}}(\boldsymbol{X}) \longmapsto \boldsymbol{\mathcal{S}}(\boldsymbol{X})$, such that for each e −edge$\boldsymbol{g}: \boldsymbol{W_e} \longrightarrow \boldsymbol{W'_e}: \boldsymbol{X} \mapsto \boldsymbol{X'}$, the following diagram commutes:

$$\begin{array}{ccc}
\boldsymbol{\mathcal{T}}(\boldsymbol{X}) & \overset{\mathfrak{A}_{W_e}}{\longrightarrow} & \boldsymbol{\mathcal{S}}(\boldsymbol{X}) \\
\downarrow \ \boldsymbol{\mathcal{T}}(\boldsymbol{g}) & & \downarrow \ \boldsymbol{\mathcal{S}}(\boldsymbol{g}) \\
\boldsymbol{\mathcal{T}}(\boldsymbol{X'}) & \underset{\mathfrak{A}_{W'_e}}{\longrightarrow} & \boldsymbol{\mathcal{S}}(\boldsymbol{X'})
\end{array} .\ ■$$

Definition 3.4 Let $\boldsymbol{\mathcal{T}}: \boldsymbol{W_e} \to \boldsymbol{W'_e}$ be an e −functor. If there exists an e −functor $\boldsymbol{\mathcal{S}}: \boldsymbol{W'_e} \to \boldsymbol{W_e}$ such that there exists natural e −isomorphisms $\mathfrak{A}: \mathbb{F}_{\boldsymbol{e}}(\boldsymbol{W_e}, \boldsymbol{W_e}) \longrightarrow \mathbb{F}_{\boldsymbol{e}}(\boldsymbol{W_e}, \boldsymbol{W_e})$ and $\mathfrak{A}': \mathbb{F}_{\boldsymbol{e}}(\boldsymbol{W'_e}, \boldsymbol{W'_e}) \longrightarrow \mathbb{F}_{\boldsymbol{e}}(\boldsymbol{W'_e}, \boldsymbol{W'_e})$ with $\mathfrak{A}(\boldsymbol{\mathcal{S}} \circ \boldsymbol{\mathcal{T}}) = \boldsymbol{id_{W_e}}$ and $\mathfrak{A}'(\boldsymbol{\mathcal{T}} \circ \boldsymbol{\mathcal{S}}) = \boldsymbol{id_{W'_e}}$then we call $\boldsymbol{\mathcal{T}}$ an **$\boldsymbol{e}$ −equivalence of the category$\boldsymbol{\mathcal{W}_e}$**. ■

Let now $\boldsymbol{\mathcal{E}_{\mathcal{C}}}(\mathbb{X})$ denote the **archetypical $\boldsymbol{e}$ −category** of all pairs $\boldsymbol{x} = \big(\boldsymbol{v}, \boldsymbol{e}^{(\boldsymbol{v})}\big)$of instantaneous local manifestations of **$\boldsymbol{e}$ −nodes** and interrelated **$\boldsymbol{e}$ −edges** on an $\boldsymbol{N}$ −online archetype $\mathbb{X}$in a Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$, together with its instantaneous local manifestations of **$\boldsymbol{e}$ −morphisms$\boldsymbol{h}$** $\in \boldsymbol{hom}\big(\boldsymbol{\mathcal{E}_{\mathcal{C}}}(\mathbb{X})\big)$. Similarly, let $\boldsymbol{\mathcal{E}_{\mathcal{C}}}(\mathbb{Y})$ be the archetypical e −category of all pairs $\boldsymbol{y} = \big(\boldsymbol{u}, \boldsymbol{e}^{(\boldsymbol{u})}\big)$of instantaneous local manifestations of **$\boldsymbol{e}-$ nodes** and interrelated$e$ −edges on an $\boldsymbol{N}$ −on line archetype $\mathbb{Y}$in another Cartesian area $\boldsymbol{\widetilde{U}_1} \times \ldots \times \boldsymbol{\widetilde{U}_N}$, together with its instantaneous local manifestations of **$\boldsymbol{e}$ −morphisms$\boldsymbol{h}$** $\in \boldsymbol{hom}\big(\boldsymbol{\mathcal{E}_{\mathcal{C}}}(\mathbb{Y})\big)$.

Definition 2.5 An $e-$ morphism $\boldsymbol{f}: \mathcal{E}_{\mathcal{C}}(\mathbb{X}) \to \mathcal{E}_{\mathcal{C}}(\mathbb{Y})$ is an **archetypical $\boldsymbol{e}-$ isomorphism** between $e-$manifestations $\boldsymbol{x} \in \mathbb{G}_{\boldsymbol{e}}(\mathbb{X})$ and $\boldsymbol{y} \in \mathcal{E}_{\mathcal{C}}(\mathbb{Y})$ if there is an $e-$ morphism $\boldsymbol{g}: \mathcal{E}_{\mathcal{C}}(\mathbb{Y}) \to \mathcal{E}_{\mathcal{C}}(\mathbb{X})$ such that $\boldsymbol{g} \circ \boldsymbol{f} = \boldsymbol{id}_{\boldsymbol{x}}$ (:$(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x}) = \boldsymbol{x}$) and $\boldsymbol{f} \circ \boldsymbol{g} = \boldsymbol{id}_{\boldsymbol{y}}$ (:$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{y}) = \boldsymbol{y}$).■

A **covariant archetypical $\boldsymbol{e}-$functor** (respectively, **contravariant archetypical $\boldsymbol{e}-$functor**) is a map $\mathcal{T}: \mathcal{E}_{\mathcal{C}}(\mathbb{X}) \to \mathcal{E}_{\mathcal{C}}(\mathbb{Y})$assigning

- to each instantaneous local $e-$manifestation $\boldsymbol{x} = \left(\boldsymbol{v}, \boldsymbol{e}^{(\boldsymbol{v})}\right)$of an $\boldsymbol{N}-$onlinearchetype$\mathbb{X}$over a Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$, an instantaneous local $e-$ manifestation $\boldsymbol{y} = \left(\boldsymbol{u}, \boldsymbol{e}^{(\boldsymbol{u})}\right)$ of an $\boldsymbol{N}-$onlinearchetype$\mathbb{Y}$over another Cartesian area $\boldsymbol{V_1} \times \ldots \times \boldsymbol{V_N}$ and
- to each $e-$morphism $\boldsymbol{f} \in \boldsymbol{hom}(\boldsymbol{x}, \boldsymbol{y})$ in $\mathcal{E}_{\mathcal{C}}(\mathbb{X})$, an $e-$morphism $\mathcal{T}(\boldsymbol{f}) \in \boldsymbol{hom}\big(\boldsymbol{T}(\boldsymbol{x}), \boldsymbol{T}(\boldsymbol{y})\big)$ in $\mathcal{E}_{\mathcal{C}}(\mathbb{Y})$ such that $\mathcal{T}(\boldsymbol{id}_{\boldsymbol{x}}) = \boldsymbol{id}_{\boldsymbol{T}(\boldsymbol{x})}$ and $\mathcal{T}(\boldsymbol{g} \circ \boldsymbol{f}) = \mathcal{T}(\boldsymbol{f}) \circ \mathcal{T}(\boldsymbol{g})$ (respectively, $\mathcal{T}(\boldsymbol{g} \circ \boldsymbol{f}) = \boldsymbol{T}(\boldsymbol{g}) \circ \boldsymbol{T}(\boldsymbol{f})$).

Definition 2.6 Let $\boldsymbol{\mathcal{A}}$ be the category of all $e-$covariantarchetypicalfunctors$\mathcal{T}: \mathcal{E}_{\mathcal{C}}(\mathbb{X}) \to \mathcal{E}_{\mathcal{C}}(\mathbb{Y})$. A **natural archetypical $\boldsymbol{e}-$transformation** $\mathfrak{a}: \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{A}}: \mathcal{T} \longmapsto \mathcal{S}$also called **archetypical$\boldsymbol{e}-$morphism of $\boldsymbol{e}-$functors**, between archetypical $e-$functors $\boldsymbol{T}, \boldsymbol{S}: \mathcal{E}_{\mathcal{C}}(\mathbb{X}) \to \mathcal{E}_{\mathcal{C}}(\mathbb{Y})$ associates to each instantaneous local manifestation $\boldsymbol{x} = \left(\boldsymbol{v}, \boldsymbol{e}^{(\boldsymbol{v})}\right) \in \mathcal{E}_{\mathcal{C}}(\mathbb{X})$of an $e-$ nodeand interrelated$e-$edge on a$\boldsymbol{N}-$onlinearchetype$\mathbb{X}$in the Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$ an $e-$morphism $\mathfrak{a}_{\boldsymbol{x}}: \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{A}}: \mathcal{T}(\boldsymbol{x}) \longmapsto \mathcal{S}(\boldsymbol{x})$ in $\mathcal{E}_{\mathcal{C}}(\mathbb{Y})$, such that for every $e-$edge$\boldsymbol{g}: \mathcal{E}_{\mathcal{C}}(\mathbb{X}) \longrightarrow \mathcal{E}_{\mathcal{C}}(\mathbb{X}): \boldsymbol{x} \mapsto \boldsymbol{x}'$ in $\mathcal{E}_{\mathcal{C}}(\mathbb{X})$ the following diagram is commutative:

$$\begin{tikzcd} \mathcal{T}(\boldsymbol{x}) \arrow[r, "\mathfrak{a}_{\boldsymbol{x}}"] \arrow[d, "\mathcal{T}(\boldsymbol{g})"] & \mathcal{S}(\boldsymbol{x}) \arrow[d, "\mathcal{S}(\boldsymbol{g})"] \\ \mathcal{T}(\boldsymbol{x}') \arrow[r, "\mathfrak{a}_{\boldsymbol{x}'}"'] & \mathcal{S}(\boldsymbol{x}') \end{tikzcd} .■$$

Definition2.7 Let $\mathcal{T}: \mathcal{E}_{\mathcal{C}}(\mathbb{X}) \to \mathcal{E}_{\mathcal{C}}(\mathbb{Y})$ be an archetypical $e-$functor between $e-$categories. If there exists an archetypical $e-$functor $\mathcal{S}: \mathcal{E}_{\mathcal{C}}(\mathbb{Y}) \to \mathcal{E}_{\mathcal{C}}(\mathbb{X})$ such that there exist two archetypical natural $e-$isomorphisms $\mathfrak{a}: \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{A}}$ and $\mathfrak{a}':: \boldsymbol{\mathcal{A}} \longrightarrow \boldsymbol{\mathcal{A}}$ with $\mathfrak{a}(\mathcal{T} \circ \mathcal{S}) = \boldsymbol{id}_{\mathcal{E}_{\mathcal{C}}(\mathbb{Y})}$ and $\mathfrak{a}'(\mathcal{S} \circ \boldsymbol{T}) = \boldsymbol{id}_{\mathcal{E}_{\mathcal{C}}(\mathbb{X})}$then we call $\mathcal{T}$ an **$\boldsymbol{e}-$equivalence between the archetypical $\boldsymbol{e}-$categories** $\mathcal{E}_{\mathcal{C}}(\mathbb{X})$ and $\mathcal{E}_{\mathcal{C}}(\mathbb{Y})$. ■

4 $\boldsymbol{e}-$products

Definition 4.1 An $e-$manifestation $\boldsymbol{X} \in \boldsymbol{W_e}$ is called **$\boldsymbol{e}-$universally attracting** if for each $\boldsymbol{Y} \in \boldsymbol{W_e}$ there exists a unique $e-$morphism of $\boldsymbol{Y}$ into $\boldsymbol{X}$. A $\boldsymbol{X} \in \boldsymbol{W_e}$ is called **$\boldsymbol{e}-$universally repelling** if for each $\boldsymbol{Y} \in \boldsymbol{W_e}$ there exists a $e-$unique morphism of $\boldsymbol{X}$ into $\boldsymbol{Y}$. ■

In what follows, we will give an example of $e-$universality in a categorical construction that will be used later to give an application of the main result. One of the more basic constructions is the $e-$product.

Definition 4.2 Let $\boldsymbol{X} \in \boldsymbol{W_e}, \boldsymbol{Y} \in \boldsymbol{W_e}$. The **$\boldsymbol{e}-$product** (or**$\boldsymbol{e}-$direct product**) $\boldsymbol{X} \sqcap \boldsymbol{Y}$ of $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\boldsymbol{W_e}$ is a triple $(\boldsymbol{Z}, \boldsymbol{f}, \boldsymbol{g})$ with $\boldsymbol{Z} \in \boldsymbol{W_e}$ and $\boldsymbol{f}: \boldsymbol{Z} \longmapsto \boldsymbol{X}$ and $\boldsymbol{g}: \boldsymbol{Z} \mapsto \boldsymbol{Y}$ two morphisms such that when given any two morphisms $\boldsymbol{\varphi}: \boldsymbol{P} \longmapsto \boldsymbol{X}$ and $\boldsymbol{\psi}: \boldsymbol{P} \mapsto \boldsymbol{Y}$, there exists a *unique* morphism $\boldsymbol{h}: \boldsymbol{P} \mapsto \boldsymbol{Z}$ making the following diagram commute:

$$\begin{tikzcd} & \boldsymbol{P} \arrow[dl, "\boldsymbol{\varphi}"'] \arrow[d, "\exists!\,\boldsymbol{h}"] \arrow[dr, "\boldsymbol{\psi}"] & \\ \boldsymbol{X} & \boldsymbol{Z} \arrow[l, "\boldsymbol{f}"] \arrow[r, "\boldsymbol{g}"'] & \boldsymbol{Y} \end{tikzcd} ■$$

Remark 4.1 This last property can be described by saying that the product has an **e −universal property**. It also tells us that the product is uniquely defined up to a unique isomorphism. ■

Remark 4.2 The above definition naturally generalizes to products $\prod_{t\in\mathbb{I}} \boldsymbol{X_t}$ over a family of e −manifestations $(\boldsymbol{X_t} \in \boldsymbol{W_e}, \boldsymbol{t} \in \mathbb{I})$. ■

Example 4.1 (e −product in the e −set category) If we take $\boldsymbol{W_e} = \boldsymbol{e_{Set}}$ and $\boldsymbol{X} = \boldsymbol{K}$, $\boldsymbol{Y} = \boldsymbol{M}$, then by setting$\boldsymbol{Z} = \boldsymbol{X} \times \boldsymbol{Y} = \boldsymbol{K} \times \boldsymbol{M}$and letting

$\boldsymbol{f} \equiv \boldsymbol{\pi_1} : \boldsymbol{Z} \longmapsto \boldsymbol{X}$ ($\Leftrightarrow \boldsymbol{\pi_1} : \boldsymbol{K} \times \boldsymbol{M} \longmapsto \boldsymbol{K}$), $\boldsymbol{g} \equiv \boldsymbol{\pi_2}: \boldsymbol{Z} \mapsto \boldsymbol{Y}$($\Leftrightarrow \boldsymbol{\pi_2} : \boldsymbol{K} \times \boldsymbol{M} \longmapsto \boldsymbol{M}$) be the projections on the first and second coordinates respectively, then

$$\boldsymbol{X} \sqcap \boldsymbol{Y} = (\boldsymbol{Z}, \boldsymbol{f}, \boldsymbol{g}) = (\boldsymbol{K} \times \boldsymbol{M}, \boldsymbol{\pi_1}, \boldsymbol{\pi_2})$$

is the e −product in the set e −category $\boldsymbol{e_{Set}}$. Indeed, let $\boldsymbol{\varphi}: \boldsymbol{\Lambda} \longmapsto \boldsymbol{K}$ and $\boldsymbol{\psi}: \boldsymbol{\Lambda} \mapsto \boldsymbol{M}$ be two e −morphisms ($\boldsymbol{\Lambda}$ is in $\boldsymbol{e_{Set}}$). It is clear that the e −morphism

$$\boldsymbol{h}: \boldsymbol{\Lambda} \mapsto \boldsymbol{K} \times \boldsymbol{M}: \boldsymbol{\lambda} \mapsto \boldsymbol{\varphi}(\boldsymbol{\lambda}) \times \boldsymbol{\psi}(\boldsymbol{\lambda})$$

satisfies the requirements of the above definition. This e −morphism is unique since its image in the first coordinate is given by $\boldsymbol{f} \equiv \boldsymbol{\pi_1}$ and its image in the second coordinate is given by$\boldsymbol{g} \equiv \boldsymbol{\pi_2}$. ■

Proposition 4.1 The $e-$ product $\boldsymbol{X} \sqcap \boldsymbol{Y}$ of $\boldsymbol{X} \in \boldsymbol{W_e}$ and $\boldsymbol{Y} \in \boldsymbol{W_e}$ is e −universally attracting in the e −category $\boldsymbol{\mathcal{D}}$ that has

- e −objects the pairs $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ of e −morphisms
$$\boldsymbol{\varphi}: \boldsymbol{W_e} \to \boldsymbol{W_e}: \boldsymbol{P} \longmapsto \boldsymbol{\varphi}(\boldsymbol{P}) = \boldsymbol{X} \text{ and } \boldsymbol{\psi}: \boldsymbol{W_e} \to \boldsymbol{W_e}: \boldsymbol{P} \mapsto \boldsymbol{\psi}(\boldsymbol{P}) = \boldsymbol{Y},$$
- e −morphisms the mappings $\boldsymbol{h}: \boldsymbol{P} \mapsto \boldsymbol{P'}$ in $\boldsymbol{W_e}$ making the diagram

$$\begin{tikzcd}
 & \boldsymbol{P} \arrow[dl, "\boldsymbol{\varphi}"'] \arrow[d, "\boldsymbol{h}"] \arrow[dr, "\boldsymbol{\psi}"] & \\
\boldsymbol{X} & \boldsymbol{P'} \arrow[l, "\boldsymbol{f}"] \arrow[r, "\boldsymbol{g}"'] & \boldsymbol{Y}
\end{tikzcd}$$

commute as the e −morphisms from the pairs $(\boldsymbol{\varphi}, \boldsymbol{\psi})$ to $(\boldsymbol{f}, \boldsymbol{g})$. ■

Whenever we reverse all edges in the graph representing an e −category$\boldsymbol{W_e}$, we obtain the graph representing the **e −dualcategory** of $\boldsymbol{W_e}$.

Definition 4.3 Let $\boldsymbol{X} \in \boldsymbol{W_e}$, $\boldsymbol{Y} \in \boldsymbol{W_e}$. The **$e$ −coproduct$\boldsymbol{X} \sqcup \boldsymbol{Y}$** of $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\boldsymbol{W_e}$ is a triple $(\boldsymbol{Z}, \boldsymbol{f}, \boldsymbol{g})$ with $\boldsymbol{Z} \in \boldsymbol{W_e}$ and $\boldsymbol{f} : \boldsymbol{X} \longmapsto \boldsymbol{Z}$ and $\boldsymbol{g}: \boldsymbol{Y} \mapsto \boldsymbol{Z}$ two morphisms such that when given any two morphisms $\boldsymbol{\varphi}: \boldsymbol{X} \longmapsto \boldsymbol{P}$ and $\boldsymbol{\psi}: \boldsymbol{Y} \mapsto \boldsymbol{P}$, there exists a *unique* morphism $\boldsymbol{h}: \boldsymbol{Z} \mapsto \boldsymbol{P}$ making the following diagram commute:

$$\begin{tikzcd}
 & \boldsymbol{P} & \\
\boldsymbol{X} \arrow[ur, "\boldsymbol{\varphi}"] \arrow[r, "\boldsymbol{f}"'] & \boldsymbol{Z} \arrow[u, "\exists!\, \boldsymbol{h}"'] & \boldsymbol{Y} \arrow[ul, "\boldsymbol{\psi}"'] \arrow[l, "\boldsymbol{g}"]
\end{tikzcd}$$

■

Proposition 4.2 (The e −coproduct in the e −set category) If we take $\boldsymbol{W_e} = \boldsymbol{e_{Set}}$ and $\boldsymbol{X} = \boldsymbol{K}$, $\boldsymbol{Y} = \boldsymbol{M}$ are two sets in $\boldsymbol{\mathcal{E}_{\mathcal{C}}}$, then by setting $\boldsymbol{Z} = \boldsymbol{X} \cup \boldsymbol{Y} =$

$\boldsymbol{K \cup M}$ and letting $\boldsymbol{f \equiv i_1 : X \longmapsto Z}$ $(\Leftrightarrow \boldsymbol{i_1 : K \hookrightarrow K \cup M})$ and $\boldsymbol{g \equiv i_2 : Z \mapsto Y}$ $(\Leftrightarrow \boldsymbol{i_2 : M \hookrightarrow K \cup M})$ be the two inclusions, then

$$\boldsymbol{X \sqcup Y = (z, i_1, i_2)}$$

is the e −coproduct in the e −set category $\boldsymbol{e_{Set}}$. The required *unique* e −morphism $\boldsymbol{h : K \cup M \longmapsto \Lambda}$ that makes the diagram

$$\begin{tikzcd} & \boldsymbol{\Lambda} & \\ \boldsymbol{X = K} \arrow[ur, "\boldsymbol{\varphi}"] \arrow[r, "\boldsymbol{f = i_1}"'] & \boldsymbol{Z = K \cup M} \arrow[u, "\exists! \boldsymbol{h}"'] & \boldsymbol{Y = M} \arrow[ul, "\boldsymbol{\psi}"'] \arrow[l, "\boldsymbol{g = i_2}"] \end{tikzcd}$$

commute is defined by the maps

$$\boldsymbol{\varphi : K \mapsto \varphi(\Lambda)} \text{ and } \boldsymbol{\psi : M \mapsto \psi(\Lambda)}. \ \blacksquare$$

Proposition 4.3 (The $\boldsymbol{e}$ −coproduct in the $\boldsymbol{e}$ −group category) Let $\boldsymbol{W_e = e_{Grp}}$ and $\boldsymbol{X = G}$, $\boldsymbol{Y = H}$ be two groups of $\boldsymbol{\mathcal{E}_C}$. By setting $\boldsymbol{Z = X * Y = G * H}$(: the free product of $\boldsymbol{G}$ and $\boldsymbol{H}$)and letting

$\boldsymbol{f \equiv i_1 : X \longmapsto Z}$ $(\Leftrightarrow \boldsymbol{i_1 : G \hookrightarrow G * H})$ and $\boldsymbol{g \equiv i_2 : Z \mapsto Y}$$(\Leftrightarrow \boldsymbol{i_2 : H \hookrightarrow G * H})$

be the two inclusions, then $\boldsymbol{X \sqcup Y = (z, i_1, i_2)}$is the e −coproduct in the e −group category $\boldsymbol{e_{Set}}$. The required *unique* e − morphism $\boldsymbol{h : G * H \longmapsto D}$ that makes the diagram

$$\begin{tikzcd} & \boldsymbol{D} & \\ \boldsymbol{X = G} \arrow[ur, "\boldsymbol{\varphi}"] \arrow[r, "\boldsymbol{f = i_1}"'] & \boldsymbol{Z = G * H} \arrow[u, "\exists! \boldsymbol{h}"'] & \boldsymbol{Y = H} \arrow[ul, "\boldsymbol{\psi}"'] \arrow[l, "\boldsymbol{g = i_2}"] \end{tikzcd}$$

commute is defined by the two groups homomorphisms $\boldsymbol{\varphi : G \mapsto \varphi(G) = D}$ and $\boldsymbol{\psi : H \mapsto \psi(H) = D}$.

Proof It is enough to choose the map $\boldsymbol{h : G * H \longmapsto D}$ by sending a word $\prod_i \boldsymbol{a_i} \in \boldsymbol{G * H}$ to $\prod_i \boldsymbol{f_i(a_i)} \in \boldsymbol{D}$, where

$$\boldsymbol{f_i} = \begin{cases} \boldsymbol{\varphi}, & if\ \boldsymbol{a_i}\ is\ in\ the\ chosen\ set\ of\ generators\ of \boldsymbol{G} \\ \boldsymbol{\psi}, & if\ \boldsymbol{a_i}\ is\ not\ a\ generator\ of\ \boldsymbol{G}. \end{cases}$$

Then, from this construction, it follows that the above diagram commutes. To show that such a e −morphism $\boldsymbol{h}$ is uniquely defined, let us suppose that $\boldsymbol{h' : G * H \longmapsto D}$ is another mapping that makes the diagram commute. Restriction of $\boldsymbol{h'}$ to $\boldsymbol{G}$ and then to $\boldsymbol{H}$ guarantees that $\boldsymbol{h' = h}$, which completes the proof.■

Remark 4.3 In the case of the e −topological category $\boldsymbol{e_{Top}}$, coproducts are disjoint unions with their disjoint union topologies. ■

Let $\boldsymbol{X \in W_e}$ be fixed. We consider the e − category $\boldsymbol{W_e^{(X)}}$ defined by the following two specifications.

- The e −objects of $\boldsymbol{W_e^{(X)}}$ are given by the e −edges $\boldsymbol{f : X \mapsto f(X)}$.
- The set of e − morphisms in $\boldsymbol{W_e^{(X)}}$ is the set of mappings $\boldsymbol{h : W_e^{(X)} \to W_e^{(X)} : A \mapsto B}$for which there exist two e −objects $\boldsymbol{f : W_e^{(X)} \to W_e^{(X)} : X \mapsto A}$ and $\boldsymbol{g : W_e^{(X)} \to W_e^{(X)} : X \mapsto B}$ such that the following diagram commutes:

$$\boldsymbol{X} \quad \boldsymbol{g}$$

$$\begin{tikzcd} & {} \arrow[dl, "f"'] \arrow[dr] & \\ \boldsymbol{A} \arrow[rr, "\boldsymbol{h}"] & & \boldsymbol{B} \end{tikzcd}$$ ■

Definition 4.4 Let $\boldsymbol{X} \in \boldsymbol{W_e}$. For any $\boldsymbol{f} \in \boldsymbol{W_e}$ and $\boldsymbol{g} \in \boldsymbol{W_e}$, the $\boldsymbol{X}$ **−fibered** $\boldsymbol{e}$ **−coproduct** $\boldsymbol{f} \sqcup_{\boldsymbol{X}} \boldsymbol{g}$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ in $\boldsymbol{W_e}$ is a triple $(\boldsymbol{h}, \boldsymbol{u}, \boldsymbol{v})$ with $\boldsymbol{h} \in \boldsymbol{W_e^{(X)}}$ and $\boldsymbol{u} : \boldsymbol{f} \longmapsto \boldsymbol{h}$ and $\boldsymbol{v}: \boldsymbol{g} \mapsto \boldsymbol{h}$ two $e-$morphisms such that when given any two $e-$ morphisms $\boldsymbol{\varphi}: \boldsymbol{f} \longmapsto \boldsymbol{\mathcal{p}}$ and $\boldsymbol{\psi}: \boldsymbol{g} \mapsto \boldsymbol{\mathcal{p}}$, there exists a *unique* $e-$ morphism $\boldsymbol{m}: \boldsymbol{h} \mapsto \boldsymbol{\mathcal{p}}$ making the following diagram commute:

$$\begin{tikzcd} & \boldsymbol{\mathcal{p}} & \\ \boldsymbol{f} \arrow[ur, "\boldsymbol{\varphi}"] \arrow[r, "\boldsymbol{u}"'] & \boldsymbol{h} \arrow[u, "\exists!\boldsymbol{m}"'] & \boldsymbol{g} \arrow[ul, "\boldsymbol{\psi}"'] \arrow[l, "\boldsymbol{v}"] \end{tikzcd}$$ ■

5 Vector weights on $e-$ categories

In general, a **weighted set** $\boldsymbol{X}$ consists of pairs $(\boldsymbol{x}, \boldsymbol{\mathcal{b}})$, where $\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{\mathcal{b}} \in [\boldsymbol{0}, \infty[$. The non-negative parameter $\boldsymbol{\mathcal{b}}$ depends on $\boldsymbol{x}$ and is the **weight parameter**, or simply the **weight** of $\boldsymbol{x}$ in $\boldsymbol{X}$. Similarly, a (multilayered) **weighted graph** $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$associates a **weight** (label) $\boldsymbol{a_j} \in \mathbb{R}$ with each node $\boldsymbol{j}$ of the vertex set $\boldsymbol{V}$ in (any layer of) the graph $\boldsymbol{G}$ and another **weight** (label) $\boldsymbol{b}_{[\boldsymbol{j},\boldsymbol{k}]} \in \mathbb{R}$ with each edge $[\boldsymbol{j}, \boldsymbol{k}]$ of the edge set $\boldsymbol{E}$ in (any layer of) the graph.

Weights of sets and graphs are usually real numbers. However, in both cases, the weights could be selected to be multiple weights or to belong in a vector space. Specifically, the weights may take values within a Euclidean space, so that each component of the weight to represent the numerical value of a quantity in a matrix of situations. In this direction, we can give the following.

Definition 5.1 A **set with vector weights** is a set into which each element corresponds to a vector called **vector weight**. Similarly, a **(multilayered) graph with vector weights** is a (multilayered) graph into which each node and each edge corresponds to a vector called, again, **vector weight**. ■

Let's give a first example, by investigating vector weights on the $e-$graph category. It is easily seen that the set $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_C})$ of all e—objects of $\boldsymbol{\mathcal{E}_C}$, together with the collection $\wp\big(\boldsymbol{ob}(\boldsymbol{\mathcal{E}_C})\big) \equiv \boldsymbol{2}^{\boldsymbol{ob}(\mathcal{E}_C)}$ of its subsets, is a **measurable space**. In fact, whenever $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \in \boldsymbol{\mathcal{S}}_{\boldsymbol{ob}(\mathcal{E}_C)}$, the set $\boldsymbol{\mathcal{X}} \setminus \boldsymbol{\mathcal{Y}}$ is in $\wp\big(\boldsymbol{ob}(\boldsymbol{\mathcal{E}_C})\big)$; and, for any $\boldsymbol{\mathcal{X}_1}, \boldsymbol{\mathcal{X}_2}, \ldots \in$

$\boldsymbol{\mathcal{S}_{ob(\mathbb{G}_e)}}$, the union $\bigcup_i \boldsymbol{\mathfrak{X}_i}$ is also in $\wp\big(\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})\big)$. Similarly, for any $\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}} \in \wp\big(\boldsymbol{hom}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})\big) \equiv \boldsymbol{2}^{\boldsymbol{hom}(\mathcal{E}_{\mathcal{C}})}$, the set $\boldsymbol{\mathfrak{X}} \setminus \boldsymbol{\mathfrak{Y}}$ is in $\wp\big(\boldsymbol{hom}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})\big)$; and, for any $\boldsymbol{\mathfrak{X}_1}$, $\boldsymbol{\mathfrak{X}_2},\ldots \in \wp\big(\boldsymbol{hom}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})\big)$, the union $\bigcup_i \boldsymbol{\mathfrak{X}_i}$ is also in $\boldsymbol{\mathcal{S}_{hom(\mathcal{E}_{\mathcal{C}})}}$. It follows that the set $\boldsymbol{hom}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$, together with the collection $\wp\big(\boldsymbol{hom}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})\big)$ of its subsets, is also a **measurable space**. Accordingly, below, we will assume that the measurable spaces $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$ and $\boldsymbol{hom}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$ are endowed with two measures, say $\boldsymbol{\mu_{ob(\mathcal{E}_{\mathcal{C}})}}$ and $\boldsymbol{\mu_{hom(\mathcal{E}_{\mathcal{C}})}}$, respectively.

Definition 5.2 i. Let $\boldsymbol{X} = (\boldsymbol{V} = \boldsymbol{V_X}, \boldsymbol{E} = \boldsymbol{E_X}) \in \boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$ be an *e*—object on $\boldsymbol{\mathcal{E}_{\mathcal{C}}}$.

a. The **maximum weight $\boldsymbol{\mathscr{b}}^{(\infty)}(\boldsymbol{X})$ of $\boldsymbol{X}$** is defined to be an ordered pair in $\mathbb{R}_+^{\boldsymbol{2}}$with

- first component is equal to the sum$[\![\boldsymbol{V_X}]\!] := \sum_{\boldsymbol{i} \in \boldsymbol{V_X}} \boldsymbol{a_i}(\boldsymbol{X})$of all weights $\boldsymbol{a_i}(\boldsymbol{X}) > \boldsymbol{0}$ at the nodes of $\boldsymbol{V_X}$ and
- second component is equal to the maximum$[\![\boldsymbol{E_X}]\!]_\infty := \boldsymbol{max}_{[\boldsymbol{j},\boldsymbol{k}] \in \boldsymbol{E_X}} \big|\boldsymbol{b}_{[\boldsymbol{j},\boldsymbol{k}]}(\boldsymbol{X})\big|$over all absolute values of weights $\boldsymbol{b}_{[\boldsymbol{j},\boldsymbol{k}]}(\boldsymbol{X}) \in \mathbb{R}$ associated with each edge $[\boldsymbol{j}, \boldsymbol{k}] \in \boldsymbol{E_X}$.

So, the **maximum weight $\boldsymbol{\mathscr{b}}^{(\infty)}(\boldsymbol{X})$ of $\boldsymbol{X}$** is equal to $\boldsymbol{\mathscr{b}}^{(\infty)}(\boldsymbol{X}) = \big([\![\boldsymbol{V_X}]\!], [\![\boldsymbol{E_X}]\!]_\infty\big)$.Based on this definition, we can define the **maximum weight $\boldsymbol{\mathscr{b}}^{(\infty)}(\boldsymbol{\mathcal{A}})$ over an entire subset$\boldsymbol{\mathcal{A}}$ of the class $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$**as

$$\boldsymbol{\mathscr{b}}^{(\infty)}(\boldsymbol{\mathcal{A}}) \equiv \int_{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathscr{b}}^{(\infty)}(\boldsymbol{X}) \boldsymbol{d\mu_{ob(\mathcal{E}_{\mathcal{C}})}}(\boldsymbol{X}) := \int_{\boldsymbol{\mathcal{A}}} \big\{[\![\boldsymbol{V_X}]\!] + \big[\![[\![\boldsymbol{E_X}]\!]_\infty \big\} \boldsymbol{d\mu_{ob(\mathcal{E}_{\mathcal{C}})}}(\boldsymbol{X}).$$

b. Analogously, the **square weight $\boldsymbol{\mathscr{b}}^{(\boldsymbol{2})}(\boldsymbol{X})$ of $\boldsymbol{X}$** is defined to be an ordered pair in $\mathbb{R}_+^{\boldsymbol{2}}$with

- first component, once again, is equal to the sum
$$[\![\boldsymbol{V_X}]\!] := \sum_{\boldsymbol{i} \in \boldsymbol{V_X}} \boldsymbol{a_i}(\boldsymbol{X})$$
- second component is equal to the square root
$$[\![\boldsymbol{E_X}]\!]_{\boldsymbol{2}} := \left(\sum_{[\boldsymbol{i},\boldsymbol{j}] \in \boldsymbol{E_X}} \frac{\boldsymbol{a_i}(\boldsymbol{X}) \boldsymbol{a_j}(\boldsymbol{X})}{[\![\boldsymbol{V_X}]\!]^{\boldsymbol{2}}} \boldsymbol{b}_{[\boldsymbol{i},\boldsymbol{j}]}^{\boldsymbol{2}}(\boldsymbol{X})\right)^{\boldsymbol{1/2}}.$$

So, the **square weight $\boldsymbol{\mathscr{b}}^{(\boldsymbol{2})}(\boldsymbol{X})$ of $\boldsymbol{X}$** equals

$$\boldsymbol{\mathscr{b}}^{(\boldsymbol{2})}(\boldsymbol{X}) = \big([\![\boldsymbol{V_X}]\!], [\![\boldsymbol{E_X}]\!]_{\boldsymbol{2}}\big).$$

Based on this definition, we define the **square weight $\boldsymbol{\mathscr{b}}^{(\boldsymbol{2})}(\boldsymbol{\mathcal{A}})$ over an entire subset $\boldsymbol{\mathcal{A}}$ of the class $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$**as

$$\boldsymbol{\mathscr{b}}^{(\boldsymbol{2})}(\boldsymbol{\mathcal{A}}) \equiv \int_{\boldsymbol{\mathcal{A}}} \boldsymbol{\mathscr{b}}^{(\boldsymbol{2})}(\boldsymbol{X}) \boldsymbol{d\mu_{ob(\mathcal{E}_{\mathcal{C}})}}(\boldsymbol{X}) := \int_{\boldsymbol{\mathcal{A}}} \big\{[\![\boldsymbol{V_X}]\!] + [\![\boldsymbol{E_X}]\!]_{\boldsymbol{2}} \big\} \boldsymbol{d\mu_{ob(\mathcal{E}_{\mathcal{C}})}}(\boldsymbol{X}).$$

ii. Let $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Y})$ be an e—morphism on $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_C})$.

The **weight $\boldsymbol{\mathcal{B}_X(h)}$ of $\boldsymbol{h}$at an $\boldsymbol{e}$—object$\boldsymbol{X} = (\boldsymbol{V_X}, \boldsymbol{E_X})$** is defined to be the pair

$$\boldsymbol{\mathcal{B}_X(h)} := \left(\prod_{j \in V_X} \boldsymbol{a}_{\boldsymbol{h}(j)}(\boldsymbol{Y}), \prod_{[j,k] \in E_X} \boldsymbol{\beta}_{[\boldsymbol{h}(j), \boldsymbol{h}(k)]}(\boldsymbol{Y})\right).$$

It follows that the **total weight $\boldsymbol{\mathcal{B}_{\mathcal{A}}(h)}$ of $\boldsymbol{h}$ over an entire subset $\boldsymbol{\mathcal{A}}$ of the class $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_C})$**equals

$$\begin{aligned}\boldsymbol{\mathcal{B}_{\mathcal{A}}(h)} &\equiv \int_{\mathcal{A}} \boldsymbol{\mathcal{B}_X(h)} d\boldsymbol{\mu}_{\boldsymbol{ob}(\mathcal{E}_C)}(\boldsymbol{X}) \\ &:= \int_{\mathcal{A}} \left\{\prod_{j \in V_X} \boldsymbol{a}_{\boldsymbol{h}(j)}(\boldsymbol{Y}) + \prod_{[j,k] \in E_X} \boldsymbol{\beta}_{[\boldsymbol{h}(j), \boldsymbol{h}(k)]}(\boldsymbol{Y})\right\} d\boldsymbol{\mu}_{\boldsymbol{ob}(\mathcal{E}_C)}(\boldsymbol{X}).\end{aligned}$$

Moreover, the **total weight $\boldsymbol{\mathcal{B}_{\mathcal{X}}(\mathfrak{X})}$ over a subset $\boldsymbol{\mathfrak{X}}$ of the class $\boldsymbol{hom}(\boldsymbol{\mathcal{E}_C})$ of $\boldsymbol{e}$—morphismsat a subset $\boldsymbol{\mathcal{X}}$ of the class $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_C})$** is equal to

$$\int_{\mathfrak{X}} \left[\int_{\mathcal{X}} \boldsymbol{\mathcal{B}_X(h)}\, d\boldsymbol{\mu}_{\boldsymbol{ob}(\mathcal{E}_C)}(\boldsymbol{X})\right] d\boldsymbol{\mu}_{\boldsymbol{hom}(\mathcal{E}_C)}(\boldsymbol{h}). \blacksquare$$

We are now in position to allocate vector weights on the objects and morphisms of other $e-$ categories. To this end, recall that if $\boldsymbol{W_e} = \boldsymbol{W_e}(\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}) \in \boldsymbol{\mathcal{W}_e} = \{\boldsymbol{e_{Set}}, \boldsymbol{e_{Gpr}}, \boldsymbol{e_{Top}}\}$ is such an $e-$ category, then $\boldsymbol{W_e}$ consists of sets of $e-$manifestations $\boldsymbol{X} = \boldsymbol{\mathcal{M}} \in \boldsymbol{W_e}$, that is sets of pairs $(\boldsymbol{V_X}, \boldsymbol{E_X})$ of instantaneous local manifestations of $e-$ nodes and interrelated $e-$edges on an $\boldsymbol{N}-$cyberarchetype germin the Cartesian area $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}$, together with all instantaneous local manifestations of $e-$morphisms $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{W_e})$. It is easy to verify the following elementary result.

Definition 5.3 If $\boldsymbol{\mathcal{M}} \in \boldsymbol{ob}(\boldsymbol{W_e})$ is any e—object on $\boldsymbol{W_e}$ and $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{W_e})$ is any $e-$morphism on $\boldsymbol{W_e}$, then the **maximum weight of $\boldsymbol{\mathcal{M}}$** is equal to the maximum weight $\boldsymbol{\mathscr{b}}^{(\infty)}(\boldsymbol{\mathcal{M}})$ over the set $\boldsymbol{\mathcal{M}}$:

$$\int_{\mathcal{M}} \{[\![\boldsymbol{V_X}]\!] + [\![\boldsymbol{E_X}]\!]_{\infty}\} d\boldsymbol{\mu}_{\boldsymbol{ob}(\mathcal{E}_C)}(\boldsymbol{X}).$$

Similarly, the **square weight of $\boldsymbol{\mathcal{M}}$** is equal to the square weight $\boldsymbol{\mathscr{b}}^{(\boldsymbol{2})}(\boldsymbol{\mathcal{M}})$ over the set $\boldsymbol{\mathcal{M}}$:

$$\int_{\mathcal{M}} \{[\![\boldsymbol{V_X}]\!] + [\![\boldsymbol{E_X}]\!]_{\boldsymbol{2}}\} d\boldsymbol{\mu}_{\boldsymbol{ob}(\mathcal{E}_C)}(\boldsymbol{X}).$$

Further, the **weight of $\boldsymbol{h}$ at $\boldsymbol{\mathcal{M}}$** is

$$\begin{aligned}\boldsymbol{\mathcal{B}_{\mathcal{M}}(h)} &= \int_{\mathcal{M}} \boldsymbol{\mathcal{B}_X(h)} d\boldsymbol{\mu}_{\boldsymbol{ob}(\mathcal{E}_C)}(\boldsymbol{X}) \\ &:= \int_{\mathcal{M}} \left\{\prod_{j \in V_X} \boldsymbol{a}_{\boldsymbol{h}(j)}(\boldsymbol{Y}) + \prod_{[j,k] \in E_X} \boldsymbol{\beta}_{[\boldsymbol{h}(j), \boldsymbol{h}(k)]}(\boldsymbol{Y})\right\} d\boldsymbol{\mu}_{\boldsymbol{ob}(\mathcal{E}_C)}(\boldsymbol{X}). \blacksquare\end{aligned}$$

It follows that **the weight of $\boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{W_e})$ over an entire subset $\boldsymbol{\mathcal{X}}$ of $\boldsymbol{W_e}$ can be defined to be the total weight $\boldsymbol{\mathcal{B}_{\mathcal{X}}(h)}$ of $\boldsymbol{h}$ over an entire subset $\boldsymbol{\mathcal{X}}$ of $\boldsymbol{W_e}$**. In particular, **the weight $\boldsymbol{\mathcal{B}_{W_e}(h)}$ of $\boldsymbol{h}$ (over the whole set $\boldsymbol{W_e}$) coincides with the total weight $\boldsymbol{\mathcal{B}_{ob(\mathcal{E_C})}(h)}$ of $\boldsymbol{h}$ over the class $\boldsymbol{ob(\mathcal{E_C})}$**:

$$\begin{aligned}\boldsymbol{\mathcal{B}_{ob(\mathcal{E_C})}(h)} &= \int_{ob(\mathcal{E_C})} \boldsymbol{\mathcal{B}_X(h)} d\boldsymbol{\mu_{ob(\mathcal{E_C})}(X)} \\ &:= \int_{ob(\mathcal{E_C})}\left\{\prod_{j\in V_X} \boldsymbol{a_{h(j)}(Y)} + \prod_{[j,k]\in E_X} \boldsymbol{\beta_{[h(j),h(k)]}(Y)}\right\} d\boldsymbol{\mu_{ob(\mathcal{E_C})}(X)}.\end{aligned}$$

6 e −Graphs, vector weights and kernels

Any e −category $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e} = \{\boldsymbol{\mathcal{E_C}}, \boldsymbol{e_{Set}}, \boldsymbol{e_{Hom}}, \boldsymbol{e_{Gpr}}, \boldsymbol{e_{Top}}\}$ can be viewed as an **e − graph** $(\mathbb{V}, \mathbb{E})$ with **vector weights**: the **e − nodes** in $\mathbb{V}$ are the e − objects $\boldsymbol{X} \in \boldsymbol{ob(W_e)}$, while the **$e$ −edges** or **e − arcs** in $\mathbb{E}$ are the e −morphisms $\boldsymbol{h} \in \boldsymbol{hom(W_e)}$, together with

- an identity map $\boldsymbol{id}: \boldsymbol{ob(W_e)} \to \boldsymbol{hom(W_e)}: \boldsymbol{X} \mapsto \boldsymbol{id_X}$ (here $\boldsymbol{id_X X = X}$) and
- a composition map $\circ: \{(\boldsymbol{g}, \boldsymbol{f}) \in \boldsymbol{hom(W_e)} \times \boldsymbol{hom(W_e)}: \boldsymbol{domg = ranf}\} \to \boldsymbol{hom(W_e)}: (\boldsymbol{g}, \boldsymbol{f}) \longmapsto \boldsymbol{g} \circ \boldsymbol{f}$[2]

such that

- $\boldsymbol{dom(id_X)} = \boldsymbol{ran(id_X)}$, whenever $\boldsymbol{X} \in \mathbb{V} \equiv \boldsymbol{ob(W_e)}$,
- $\boldsymbol{dom(g \circ f)} = \boldsymbol{dom(f)}$ and $\boldsymbol{ran(g \circ f)} = \boldsymbol{ran(g)}$, for all $\boldsymbol{g}, \boldsymbol{f} \in \boldsymbol{hom(W_e)}$ with $\boldsymbol{domg = ranf}$,
- $\boldsymbol{id_x \circ f = f \circ id_Y}$ holds for all $\boldsymbol{f} \in \boldsymbol{hom(X, Y)}$ ($\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{V} \equiv \boldsymbol{ob(W_e)}$),
- $(\boldsymbol{f} \circ \boldsymbol{g}) \circ \boldsymbol{h} = \boldsymbol{f} \circ (\boldsymbol{g} \circ \boldsymbol{h})$ for pairs and edges with configuration

$$\boldsymbol{X} \xrightarrow{\boldsymbol{h}} \boldsymbol{Y} \xrightarrow{\boldsymbol{g}} \boldsymbol{Z} \xrightarrow{\boldsymbol{f}} \boldsymbol{U}.$$

Obviously, an e − morphism $\boldsymbol{f} \in \boldsymbol{hom(X, Y)}$ of the e − category $\boldsymbol{W_e}$ corresponds to the e −edge $[\boldsymbol{X}, \boldsymbol{Y}]$ that starts at the e −manifestation $\boldsymbol{X}$ and ends at the e − manifestation $\boldsymbol{Y}$. Especially, for the e − graph $\boldsymbol{\mathfrak{G}} = \boldsymbol{\mathfrak{G}_{\mathcal{E_C}}}$ corresponding to the e − graph category $\boldsymbol{\mathcal{E_C}}$, the vector weight of the **e − node** associated to the e − manifestation $\boldsymbol{X} = (\boldsymbol{V_X}, \boldsymbol{E_X}) \in \mathbb{V} \equiv \boldsymbol{ob(\mathcal{E_C})}$ is equal to the **maximum weight**

[2] $\boldsymbol{domg}$ =thedomain of definition of $\boldsymbol{g}$; $\boldsymbol{ranf}$ =the range of $\boldsymbol{f}$

$\boldsymbol{\mathscr{b}^{(\infty)}(X)}$ **of** $\boldsymbol{X}$ i.e. $\mathscr{b}^{(\infty)}(X) = \left([\![V_X]\!], [\![E_X]\!]_\infty\right)$ or, alternatively, to the **square weight** $\boldsymbol{\mathscr{b}^{(2)}(X)}$ **of** $\boldsymbol{X}$ i.e. $\mathscr{b}^{(2)}(X) = \left([\![V_X]\!], [\![E_X]\!]_2\right)$.

The **maximum weight** $\boldsymbol{\mathscr{b}^{(\infty)}(\mathcal{A})}$ **of an entire subset** $\boldsymbol{\mathcal{A}}$ **of** $\boldsymbol{e-}$**nodes** associated to an entire **class of** $e-$manifestations $X = (V_X, E_X) \in ob(\mathcal{E}_C)$is equal to

$$\mathscr{b}^{(\infty)}(\mathcal{A}) \equiv \int_{\mathcal{A}} \mathscr{b}^{(\infty)}(X) d\mu_{ob(\mathcal{E}_C)}(X) := \int_{\mathcal{A}} \left\{[\![V_X]\!] + [\![E_X]\!]_\infty\right\} d\mu_{ob(\mathcal{E}_C)}(X).$$

Similarly, the **square weight** $\boldsymbol{\mathscr{b}^{(2)}(\mathcal{A})}$ **of an entire subset** $\boldsymbol{\mathcal{A}}$ **of** $\boldsymbol{e-}$**nodes** associated to an entire **class of** $e-$manifestations $X = (V_X, E_X) \in ob(\mathcal{E}_C)$is equal to

$$\mathscr{b}^{(2)}(\mathcal{A}) \equiv \int_{\mathcal{A}} \mathscr{b}^{(2)}(X) d\mu_{ob(\mathcal{E}_C)}(X) := \int_{\mathcal{A}} \left\{[\![V_X]\!] + [\![E_X]\!]_2\right\} d\mu_{ob(\mathcal{E}_C)}(X).$$

And the **weight**$\boldsymbol{\mathcal{B}[X,Y]}$**of the** $\boldsymbol{e-}$**edge**$[X,Y]$ of $\mathbb{E}$ associated to an $e-$morphism $h \in hom(X,Y)$ is the **total weight** $\boldsymbol{\mathcal{B}(h)}$ **of** $\boldsymbol{h}$ **over the entire class** $\boldsymbol{ob(\mathcal{E}_C)}$**,** i.e.

$$\mathcal{B}[X,Y] = \mathcal{B}_{ob(\mathbb{G}_e)}(h) \equiv \int_{ob(\mathcal{E}_C)} \mathcal{B}_X(h) d\mu_{ob(\mathcal{E}_C)}(Z)$$
$$:= \int_{ob(\mathcal{E}_C)} \left\{\prod_{j\in V_Z} a_{h(j)}(Y) + \prod_{[j,k]\in E_Z} \beta_{[h(j),h(k)]}(Y)\right\} d\mu_{ob(\mathcal{E}_C)}(Z).$$

Similarly, for the $e-$graph $\mathfrak{G} = \mathfrak{G}_{W_e}$ corresponding to an $e-$ category $W_e = \left\{e_{Set}, e_{Hom}, e_{Gpr}, e_{Top}\right\}$, the **vector weight of the** $\boldsymbol{e-}$**node associated to the** $\boldsymbol{e-}$**manifestation** $\boldsymbol{X = \mathcal{M} \in W_e}$ is equal to the **maximum weight** $\boldsymbol{\mathscr{b}^{(\infty)}(\mathcal{M})}$ **of** $\boldsymbol{\mathcal{M}}$, i.e.

$$\mathscr{b}^{(\infty)}(\mathcal{M}) = \int_{\mathcal{M}} \left\{[\![V_X]\!] + [\![E_X]\!]_\infty\right\} d\mu_{ob(\mathbb{G}_e)}(X)$$

or alternatively, to the **square weight** $\boldsymbol{\mathscr{b}^{(2)}(\mathcal{M})}$ **of** $\boldsymbol{\mathcal{M}}$, i.e.

$$\int_{\mathcal{M}} \left\{[\![V_X]\!] + [\![E_X]\!]_2\right\} d\mu_{ob(\mathcal{E}_C)}(X).$$

And the **weight** $\boldsymbol{\mathfrak{B}[X,Y]}$ of the $\boldsymbol{e-}$**edge** $[X,Y]$ of $\mathbb{E}$ associated to an $e-$morphism $h \in hom(X,Y)$ is the **weight** $\boldsymbol{\mathcal{B}_{W_e}(h)}$ **of** $\boldsymbol{h}$ (**over the whole set** $\boldsymbol{W_e}$) which coincides with the total weight $\mathcal{B}_{ob(\mathcal{E}_C)}(h)$ of h over the class $ob(\mathcal{E}_C)$:

$$\mathfrak{B}[X,Y] = \mathcal{B}_{W_e}(h) \equiv \int_{ob(\mathcal{E}_C)} \mathcal{B}_Z(h) d\mu_{ob(\mathcal{E}_C)}(Z)$$
$$:= \int_{ob(\mathcal{E}_C)} \left\{\prod_{j\in V_Z} a_{h(j)}(Y) + \prod_{[j,k]\in E_Z} \beta_{[h(j),h(k)]}(Y)\right\} d\mu_{ob(\mathcal{E}_C)}(Z).$$

The crucial for our purposes theoretical result that will be shown below proves that each $e-$category $W_e$is a metric space and, thus, provides a suitable topological structure on every W_e permitting profound and rigorous analytic investigation of a central new concept introduced in the sequel, the so-called "local cyber-mapping".

However, before entering into more details, let me give another special characterization of the e −graph $\mathfrak{G} = \mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}$.

Let $\mathfrak{K}$ denote the space of all bounded symmetric measurable functions $\boldsymbol{f}:[\mathbf{0},\mathbf{1}]^{\mathbf{2}} \to \mathbb{R}$. The elements of $\mathfrak{K}$ will be called **kernels** ([19]). Let also $\mathfrak{K}_{\mathbf{0}}$ denote the set of all kernels $\boldsymbol{f} \in \mathfrak{K}$ such that $\boldsymbol{f}([\mathbf{0},\mathbf{1}]^{\mathbf{2}}) \subset [\mathbf{0},\mathbf{1}]$. The elements of $\mathfrak{K}_{\mathbf{0}}$ will be called **graphons** ([19]) Sometimes we will also need to consider the set of all functions $\boldsymbol{f} \in \mathfrak{K}$ such that $\boldsymbol{f}([\mathbf{0},\mathbf{1}]^{\mathbf{2}}) \subset [-\mathbf{1},\mathbf{1}]$; this will be denoted by $\mathfrak{K}_{\mathbf{1}}$.

Kernels generalize e −graphs in the following sense. A function $\boldsymbol{f} \in \mathfrak{K}$ is called a **step function**, if there is a partition $\boldsymbol{S}_{\mathbf{1}} \cup \cdots \cup \boldsymbol{S}_{\boldsymbol{k}}$ of $[\mathbf{0},\mathbf{1}]$ into measurable sets such that $\boldsymbol{f}$ is constant on every product set $\boldsymbol{S}_{\boldsymbol{i}} \times \boldsymbol{S}_{\boldsymbol{j}}$. The sets $\boldsymbol{S}_{\boldsymbol{i}}$ are the **steps** of $\boldsymbol{f}$. For the **infinite** weighted e −graph $\mathfrak{G} = \mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}$ corresponding to the e −graph category $\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{C}}}$, we define step functions as follows.

- The $\boldsymbol{e}$ **−node** of $\mathfrak{G}$ associated to the e −manifestation $(\boldsymbol{V}_{\boldsymbol{X}}, \boldsymbol{E}_{\boldsymbol{X}}) \in \boldsymbol{ob}(\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{C}}})$ is again denoted by $(\boldsymbol{V}_{\boldsymbol{X}}, \boldsymbol{E}_{\boldsymbol{X}})$.
- Observe that, by construction, the $e-$ graph $\mathfrak{G}$ has the following obvious properties.
  - Each node of $\mathfrak{G}$ is a finite graph.
  - $\mathfrak{G}$ is an infinite graph.
  - $\mathfrak{G}$ is a locally finite graph.
  - $\mathfrak{G}$ is a connected graph. This means that there is a path (: homomorphism) between every pair of vertices of $\mathfrak{G}$. This is a not strong assumption: in fact, if a vertex $\boldsymbol{X}$ of $\mathfrak{G}$ is not connected with other vertices of $\mathfrak{G}$, then there is no e −morphism from $\boldsymbol{X}$ to any $\boldsymbol{Y}$ in $\mathfrak{G}$. Following Definition 2.1.ii, this is impossible.
- Consider the **end compactification** $|\mathfrak{G}|$ of $\mathfrak{G}$ obtained by adding the **ends** of $\mathfrak{G}$. Loosely speaking, the ends of $\mathfrak{G}$ are the "path components of $\mathfrak{G}$ at infinity". See the paper [15] where this heuristic is made precise using the language of nonstandard analysis. End compactifications of graphs have been considered in [2], [7], [9], [10], [12] and [13] as a way to obtain analogues for infinite graphs of results infinite graph theory that would otherwise be plainly false.

Remark 6.1 Ends of graphs were defined by Rudolf Halin, in 1964, in terms of equivalence classes of infinite paths (see [16]; however, in 2008, as Krön & Möller pointed out, ends of graphs were already considered by Freudenthal in [11]). A **ray** in an infinite graph is a semi-infinite simple path; that is, it is an infinite sequence of nodes $\boldsymbol{V_0}$, $\boldsymbol{V_1}$, $\boldsymbol{V_2}$, ... in which each vertex appears at most once in the sequence and each two consecutive nodes in the sequence are the two endpoints of an edge in the graph. According to Halin's definition, two rays $\boldsymbol{r_0}$ and $\boldsymbol{r_1}$ are equivalent if there is another ray $\boldsymbol{r_2}$ (not necessarily different from either of the first two rays) that contains infinitely many of the nodes in each of $\boldsymbol{r_0}$ and $\boldsymbol{r_1}$. This is an equivalence relation: each ray is equivalent to itself, the definition is symmetric with regard to the ordering of the two rays, and it can be shown to be transitive. Therefore, it partitions the set of all rays into equivalence classes, and Halin defined an end as one of these equivalence classes. An alternative definition of the same equivalence relation has also been used: two rays $\boldsymbol{r_0}$ and $\boldsymbol{r_1}$ are equivalent if there is no finite set of nodes that separates infinitely many nodes of $\boldsymbol{r_0}$ from infinitely many nodes of $\boldsymbol{r_1}$. This is equivalent to Halin's definition: if the ray $\boldsymbol{r_2}$ from Halin's definition exists, then any separator must contain infinitely many points of $\boldsymbol{r_2}$ and therefore cannot be finite, and conversely if $\boldsymbol{r_2}$ does not exist then a path that alternates as many times as possible between $\boldsymbol{r_0}$ and $\boldsymbol{r_1}$ must form the desired finite separator.■

- The **end compactification** $|\boldsymbol{\mathfrak{G}}|$ of $\boldsymbol{\mathfrak{G}}$ is a compact and metrizable topological space. (For more information we refer to the article [8]). This implies that for any infinite open cover $\boldsymbol{O_1}$, $\boldsymbol{O_2}$,… of $|\boldsymbol{\mathfrak{G}}|$, there exists a finite subcover $\boldsymbol{O_{a_1}}$,…, $\boldsymbol{O_{a_L}}$. Choose such a finite subcover $\boldsymbol{O_{a_1}}$,…, $\boldsymbol{O_{a_L}}$ of $|\boldsymbol{\mathfrak{G}}|$. It is clear that the restrictions $\boldsymbol{\Omega_1} \coloneqq \boldsymbol{O_{a_1}} \cap \boldsymbol{\mathfrak{G}}$,…, $\boldsymbol{\Omega_L} \coloneqq \boldsymbol{O_{a_L}} \cap \boldsymbol{\mathfrak{G}}$form also a finite cover of $\boldsymbol{\mathfrak{G}}$.

1st Way of introduction of a step function associated to the $\boldsymbol{e}$ −graph $\boldsymbol{\mathfrak{G}}$

- Consider the normalized maximum weights

$$\boldsymbol{\lambda_1^{(\infty)}} = \frac{\boldsymbol{\mathscr{b}^{(\infty)}(\Omega_1)}}{\boldsymbol{\mathscr{b}^{(\infty)}(\Omega_1)+\cdots+\mathscr{b}^{(\infty)}(\Omega_L)}}, \ldots, \boldsymbol{\lambda_L^{(\infty)}} = \frac{\boldsymbol{\mathscr{b}^{(\infty)}(\Omega_L)}}{\boldsymbol{\mathscr{b}^{(\infty)}(\Omega_1)+\cdots+\mathscr{b}^{(\infty)}(\Omega_L)}}.$$

- Split $[\boldsymbol{0}, \boldsymbol{1}]$ into $\boldsymbol{L}$ intervals $\boldsymbol{J_1^{(\infty)}}$, . . . , $\boldsymbol{J_L^{(\infty)}}$ of length $\boldsymbol{\lambda_1^{(\infty)}}$, … , $\boldsymbol{\lambda_L^{(\infty)}}$ respectively.

- For $\boldsymbol{x} \in \boldsymbol{J_i^{(\infty)}}$ and$\boldsymbol{y} \in \boldsymbol{J_j^{(\infty)}}$, let

$$\boldsymbol{f_{\mathfrak{G}}(x,y) = f_{\mathfrak{G}}^{(\infty)} = \mathfrak{B}[\Omega_i, \Omega_j]}.$$

2nd Way of introduction of a step function associated to the $\boldsymbol{e}$ −graph $\boldsymbol{\mathfrak{G}}$

- Consider the normalized square weights

$$\boldsymbol{\mu_1^{(2)}} = \frac{\boldsymbol{\mathscr{b}^{(2)}(\Omega_1)}}{\boldsymbol{\mathscr{b}^{(2)}(\Omega_1)+\cdots+\mathscr{b}^{(2)}(\Omega_L)}}, \ldots, \boldsymbol{\mu_L^{(2)}} = \frac{\boldsymbol{\mathscr{b}^{(2)}(\Omega_L)}}{\boldsymbol{\mathscr{b}^{(2)}(\Omega_1)+\cdots+\mathscr{b}^{(2)}(\Omega_L)}}.$$

- Split $[\mathbf{0},\mathbf{1}]$ into $\boldsymbol{L}$ intervals $\boldsymbol{I}_{\mathbf{1}}^{(\infty)}$, . . . , $\boldsymbol{I}_{\boldsymbol{L}}^{(\infty)}$ of length $\boldsymbol{\mu}_{\mathbf{1}}^{(\mathbf{2})}$, … , $\boldsymbol{\mu}_{\boldsymbol{L}}^{(\mathbf{2})}$ respectively.
- For $\boldsymbol{x} \in \boldsymbol{I}_{\boldsymbol{i}}^{(\mathbf{2})}$ and $\boldsymbol{y} \in \boldsymbol{I}_{\boldsymbol{j}}^{(\mathbf{2})}$, let $\boldsymbol{g}_{\mathfrak{G}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{g}_{\mathfrak{G}}^{(\mathbf{2})} = \mathfrak{B}[\boldsymbol{\Omega}_{\boldsymbol{i}},\boldsymbol{\Omega}_{\boldsymbol{j}}]$.

Note that both multistep functions $\boldsymbol{f}_{\mathfrak{G}}$ and $\boldsymbol{g}_{\mathfrak{G}}$ depend on how the infinite open cover $\boldsymbol{O}_{\mathbf{1}}$, $\boldsymbol{O}_{\mathbf{2}}$,… of $|\mathfrak{G}|$ is chosen.

7 Topology on e −graphs and e −categories

Fix any $e-$ graph $\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}} = (\mathbb{V},\mathbb{E})$ corresponding to the $e-$ graph category $\boldsymbol{W}_{\boldsymbol{e}} \in \boldsymbol{\mathcal{W}}_{\boldsymbol{e}} = \{\boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{C}}}, \boldsymbol{e}_{\boldsymbol{Set}}, \boldsymbol{e}_{\boldsymbol{Hom}}, \boldsymbol{e}_{\boldsymbol{Gpr}}, \boldsymbol{e}_{\boldsymbol{Top}}\}$ and the function $\boldsymbol{B}: \mathbb{E} \longrightarrow]\mathbf{0},\infty[$ assigning to each e −edge$[\boldsymbol{X},\boldsymbol{Y}] \in \mathbb{E}$ its weight

$$\boldsymbol{B}[\boldsymbol{X},\boldsymbol{Y}] = \begin{cases} \boldsymbol{\mathcal{B}}[\boldsymbol{X},\boldsymbol{Y}], & \boldsymbol{if W_e = \mathcal{E}_{\mathcal{C}}} \\ \mathfrak{B}[\boldsymbol{X},\boldsymbol{Y}], & \boldsymbol{if W_e \in \mathcal{W}_e \setminus \mathcal{E}_{\mathcal{C}}} \end{cases}.$$

Without any loss of generality, the $e-$graph $\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}$ is considered to be a 1-complex, which means that the edges of $\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}$ are homeomorphic copies of the real unit interval. A **half-edge** of $\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}$ is a connected subset of an edge of $\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}$.We can use $\boldsymbol{\mathcal{B}}$ to define a **distance function** on $\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}$. Indeed, for any $\boldsymbol{X},\boldsymbol{Y} \in \mathbb{V}$, let

$$\boldsymbol{d}_{\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}}(\boldsymbol{X},\boldsymbol{Y}) := \mathbf{inf}\{\boldsymbol{B}(\boldsymbol{P}): \boldsymbol{P\ is\ an\ X-Y\ path}\}$$

where an $\boldsymbol{X-Y\ path}$ is a finite sequence

$$\mathbb{E}(\boldsymbol{P}) = \{[\boldsymbol{X}=\boldsymbol{X}_{\mathbf{0}},\boldsymbol{X}_{\mathbf{1}}],[\boldsymbol{X}_{\mathbf{1}},\boldsymbol{X}_{\mathbf{2}}],\dots,[\boldsymbol{X}_{\boldsymbol{k}-\mathbf{1}},\boldsymbol{X}_{\boldsymbol{k}}=\boldsymbol{Y}]\}$$

of successive e −edges $\mathbb{e}$, the first e −edge $[\boldsymbol{X}_{\mathbf{0}},\boldsymbol{X}_{\mathbf{1}}]$ of this sequence having source at $\boldsymbol{X}_{\mathbf{0}} = \boldsymbol{X}$ while the last one $[\boldsymbol{X}_{\boldsymbol{k}-\mathbf{1}},\boldsymbol{X}_{\boldsymbol{k}}]$ having target at $\boldsymbol{X}_{\boldsymbol{k}} = \boldsymbol{Y}$ and where $\boldsymbol{B}(\boldsymbol{P}) \coloneqq \sum_{\mathbb{e}\in\mathbb{E}(\boldsymbol{P})} \boldsymbol{B}(\mathbb{e})$ is defined to be the **length** of the path $\boldsymbol{P}$ (relative to the distance $\boldsymbol{d}_{\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}}$).

Remark 7.1 For points that might lie in the interior of an edge, we define $\boldsymbol{d}_{\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}}(\boldsymbol{X},\boldsymbol{Y})$ similarly, but instead of graph theoretical edges we consider edges in the 1-complex $\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}$: let again

$$\boldsymbol{d}_{\mathfrak{G}_{\boldsymbol{W}_{\boldsymbol{e}}}}(\boldsymbol{X},\boldsymbol{Y}) := \mathbf{inf}\{\boldsymbol{B}(\boldsymbol{P}): \boldsymbol{P\ is\ an\ X-Y\ arc}\}$$

where $\boldsymbol{B}(\boldsymbol{P})$ is now the sum of the lengths of the edges and maximal half-edges in $\boldsymbol{P}$. Note that this sum equals the length of $\boldsymbol{P}$. ■

By identifying any two vertices $\boldsymbol{X}, \boldsymbol{X}'$ of $\mathfrak{G}_{\boldsymbol{W_e}}$ for which $\boldsymbol{d}_{\mathfrak{G}_{W_e}}(\boldsymbol{X}, \boldsymbol{X}') = \boldsymbol{0}$ holds, we obtain the metric space $(\overline{\mathfrak{G}}_{\boldsymbol{W_e}}, \boldsymbol{d}_{\boldsymbol{W_e}})$. Since $\mathfrak{G}_{\boldsymbol{W_e}}$ is locally finite, we have $\overline{\mathfrak{G}}_{\boldsymbol{W_e}} = \mathfrak{G}_{\boldsymbol{W_e}}$, and, hence

Theorem 7.1 The e −graph $\mathfrak{G}_{\boldsymbol{W_e}}$ endowed with the metric $\boldsymbol{d}_{\mathfrak{G}_{W_e}}$ is a metric space. ■

Remark 7.2 If, in particular, $\boldsymbol{W_e} = \boldsymbol{\mathcal{E}_{\mathcal{C}}}$, then the topology of $\mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}$ induced by the metric $\boldsymbol{d}_{\mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}}$ coincides with Diestel's metrizable topology described in the preceding section (see Section 3 in [14]).■

The $\boldsymbol{e}-$ graph $\mathfrak{G}_{\boldsymbol{W_e}}$ coincides with the $\boldsymbol{e}-$ category $\boldsymbol{W_e}$. Sincean $e-$ morphism $\boldsymbol{f} \in \boldsymbol{hom}(\boldsymbol{X}, \boldsymbol{Y})$ of the $e-$ category $\boldsymbol{W_e}$ is identified with the e −edge$[\boldsymbol{X}, \boldsymbol{Y}]$ that starts at the e −manifestation $\boldsymbol{X}$ and ends at the e −manifestation $\boldsymbol{Y}$, we can also view the metric $\boldsymbol{d}_{\mathfrak{G}_{W_e}}$ as a metric in the set of objects $\boldsymbol{ob}(\boldsymbol{W_e})$ of the e −category $\boldsymbol{W_e}$. So, we have the following.

Corollary 7.1 The set $\boldsymbol{ob}(\boldsymbol{W_e})$ of objects of the e −category $\boldsymbol{W_e}$ endowed with the metric $\boldsymbol{d}_{\boldsymbol{W_e}} \equiv \boldsymbol{d}_{\mathfrak{G}_{W_e}}$is a metric space. ■

8 Cyber-elements and Cyber-domains

The introduction of appropriate metric in the set $\boldsymbol{ob}(\boldsymbol{W_e})$ of objects of an$e$ −category $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e}$will allow the consideration of open, closed, compact, dense and connected areas in $\boldsymbol{ob}(\boldsymbol{W_e})$. And, as it is expected, themost significantbenefits coming from the consideration of this appropriate topology in $\boldsymbol{ob}(\boldsymbol{W_e})$can be derived fromthe definitions of the concepts of **cyber-evolution** and **cyber-domain** given below.

Definition 8.1 Let $\boldsymbol{W_e}$ be an e −category with its object set $\boldsymbol{ob}(\boldsymbol{W_e})$ endowed with the topology induced by the metric $\boldsymbol{d}_{\boldsymbol{W_e}}$.

i. A mapping $\mathcal{cy}: \mathbb{I}(\subset \mathbb{R}) \rightarrow (\boldsymbol{ob}(\boldsymbol{W_e}), \boldsymbol{d}_{\boldsymbol{W_e}})$, $\mathbb{I} = [0,1]$ = the closed unit interval in $\mathbb{R}$, is said to be **local $\boldsymbol{e}$—dynamics**.
ii. It is clear that for each $\boldsymbol{t} \in \mathbb{I}$, the corresponding image $\mathcal{cy}(\boldsymbol{t})$ has the form of an instantaneous local e −node manifestation with the interrelated$e$ −edge manifestation. The overall image $\mathcal{cy}(\mathbb{I})$ is said to be an **$\boldsymbol{e}$—arrangement**. An

e −arrangement together with all of its instant e—morphisms is called an $\boldsymbol{e}$ **−regularization**. It is denoted by $\widetilde{cy}(\mathbb{I})$.

iii. If $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N} = \boldsymbol{\Sigma_1} \times \ldots \times \boldsymbol{\Sigma_N}$, then the above mapping is called **global**$\boldsymbol{e}$ **−dynamics**. In such a case, the image $cy(\mathbb{I})$ is said to be a **global** $\boldsymbol{e}$ **−arrangement**; the associated e − regularization is called $\boldsymbol{e}$ **− settlement**.■

Remark 8.1 Definition 7.1 gives the dynamics of the local or total (information) “flow” being developed between e −nodes of the Internet, meaning that at any time there are new information flows between existing and new e-nodes which appear in the same moment that others nodes repealed.■

It is clear that in case where a local e −dynamics cy is continuous, the local e − dynamics is simply a path in the topological space $\left(\boldsymbol{ob}(\boldsymbol{W_e}), \boldsymbol{d_{W_e}}\right)$. It is absolutely essential for our purposes to consider the important class of continuous local e − dynamics **into** the **completion** $|\boldsymbol{ob}(\boldsymbol{W_e})|$ of $\boldsymbol{ob}(\boldsymbol{W_e})$ in $\overline{\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N}} \subset \mathbb{C}\mathbf{P}^{\boldsymbol{N}}$, which contains $\boldsymbol{ob}(\boldsymbol{W_e})$ as a dense subspace.

Definition 8.2 The elements of $|\boldsymbol{ob}(\boldsymbol{W_e})|$ will be called **cyber-elements**. The topological space $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$ will be called a **cyber-domain**. A continuous local e −dynamics $cy: \mathbb{I} \to \left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$, $\mathbb{I} = [0,1]$ = the closed unit interval in $\mathbb{R}$, is said to be a **cyber-evolutionary path** or simply **cyber-evolution** of the cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$. Sometimes, we may use the compellation **cyber-track** of $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$. In such a case, the e—arrangement $cy(\mathbb{I})$ is called a **cyber-arrangement**. A cyber-arrangement together with all of its instant homomorphisms is called a **cyberspace**. It is denoted by $\widetilde{cy}(\mathbb{I})$. If, in particular, $\boldsymbol{U_1} \times \ldots \times \boldsymbol{U_N} = \boldsymbol{\Sigma_1} \times \ldots \times \boldsymbol{\Sigma_N}$, then the corresponding cyber-arrangement $cy(\mathbb{I})$ is said to be a **cyber-configuration**; the associated cyberspace is also called **cyberspace**.■

9 Projective $\boldsymbol{e}$ −systems

In this Section, we will give a brief study of local e −dynamics.

Definition 9.1 Suppose $cy: \mathbb{I} \to \boldsymbol{ob}(\boldsymbol{W_e}): t \longmapsto cy(\boldsymbol{s})$ is a local e − dynamics. The e − regularization $\widetilde{cy}(\mathbb{I})$ coming from the e −arrangement $cy(\mathbb{I}) = \left(cy(\boldsymbol{s})\right)_{\boldsymbol{s} \in \mathbb{I}}$ is called a **projective e −system** on $\boldsymbol{W_e}$, if for any pair $(s,t) \in \mathbb{I}^2$ satisfying $s \leq t$, there is a e −morphism $h_s^t: cy(t) \mapsto cy(s)$ satisfying the following two conditions:

i. $\left(h_t^t = id_{cy(t)}, \forall t \in \mathbb{I}\right)$ and

ii. $(\forall s,t,r \in \mathbb{I}$ with $s \leq t \leq r \Longrightarrow h_t^s \circ h_s^r = h_t^r)$. ■

Definition 9.2 Let $\boldsymbol{W_e} = \boldsymbol{e_{Hom}}$. Suppose $cy: \mathbb{I} \to \boldsymbol{ob}(\boldsymbol{e_{Hom}}): \boldsymbol{t} \longmapsto cy(\boldsymbol{s})$ is a local e −dynamics and assume that the associated e − regularization $\widetilde{cy}(\mathbb{I})$ coming from the e −arrangement $cy(\mathbb{I})$ is a projective e −system. The **projective e −limit**

$$\lim_{\overleftarrow{t \in \mathbb{I}}} cy(t)$$

of the e −arrangement $cy(\mathbb{I})$ in $\boldsymbol{ob}(\boldsymbol{e_{Hom}})$ is defined to be the set

$$\left\{(\boldsymbol{x}(\boldsymbol{t}))_{\boldsymbol{t}\in \mathbb{I}} \in \prod_{\boldsymbol{t}\in \mathbb{I}} cy(t) : \forall s,t \in \mathbb{I} with s \leq t \Longrightarrow h_s^t\big(x(t)\big) = \big(x(s)\big)\right\}. \blacksquare$$

To each e −manifestation $\boldsymbol{A} \in \boldsymbol{W_e}$, we associate the contravariant functor $\boldsymbol{\mathcal{T}_A}: \boldsymbol{W_e} \to \boldsymbol{ob}(\boldsymbol{e_{Hom}})$ that maps all e −manifestations $\boldsymbol{Y} \in \boldsymbol{W_e}$ to the set $\boldsymbol{hom}(\boldsymbol{Y},\boldsymbol{A})$ in $\boldsymbol{ob}(\boldsymbol{e_{Hom}})$.

Definition 9.3 An e −functor $\boldsymbol{\mathcal{T}}: \boldsymbol{W_e} \longrightarrow \boldsymbol{e_{Hom}}$ is said to be **e −representable**, if there exists an e −manifestation $\boldsymbol{X}$ of $\boldsymbol{W_e}$ such that $\boldsymbol{\mathcal{T}_A}$ is naturally isomorphic to $\boldsymbol{\mathcal{T}}$ in the sense that there exists an *invertible* natural e − transformation $\mathfrak{A}_{W_e}: \boldsymbol{\mathcal{T}} \longmapsto \boldsymbol{\mathcal{T}_A}$, between the e − functors $\boldsymbol{\mathcal{T}}, \boldsymbol{\mathcal{T}_A}: \boldsymbol{W_e} \to \boldsymbol{e_{Hom}}$ which associates to each e −manifestation $\boldsymbol{X} \in \boldsymbol{W_e}$ an e −morphism $\mathfrak{A}_{\boldsymbol{X}}$: $\boldsymbol{\mathcal{T}}(\boldsymbol{X}) \longmapsto \boldsymbol{\mathcal{T}_A}(\boldsymbol{X})$, such that for each e −edge $\boldsymbol{g}: \boldsymbol{W_e} \longrightarrow \boldsymbol{e_{Hom}}: \boldsymbol{X} \mapsto \boldsymbol{X}'$, the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{\mathcal{T}}(\boldsymbol{X}) & \xrightarrow{\mathfrak{A}_{W_e}} & \boldsymbol{\mathcal{T}_A}(\boldsymbol{X}) \\ \downarrow\ \boldsymbol{\mathcal{T}}(\boldsymbol{g}) & & \downarrow\ \boldsymbol{\mathcal{T}_A}(\boldsymbol{g}). \blacksquare \\ \boldsymbol{\mathcal{T}}(\boldsymbol{X}') & \xrightarrow[\mathfrak{A}_{e_{Hom}}]{} & \boldsymbol{\mathcal{T}_A}(\boldsymbol{X}') \end{array}$$

We are now in position to generalize the last definition of the precedent Section.

Definition 9.4 Let $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e}$ be an e − category different from $\boldsymbol{e_{Hom}}$. Suppose $cy: \mathbb{I} \to \boldsymbol{ob}(\boldsymbol{W_e}): \boldsymbol{t} \longmapsto cy(\boldsymbol{s})$ is a local e −dynamicsand assume that the associated e − regularization $\widetilde{cy}$ coming from the e − arrangement $cy(\mathbb{I})$ is a projective e −system $\left(cy(\boldsymbol{s}), (h_s^t)\right)_{\boldsymbol{s} \in \mathbb{I}}$on $\boldsymbol{W_e}$. If the e −functor

$$\boldsymbol{\mathcal{T}}: \boldsymbol{W_e} \longrightarrow \boldsymbol{e_{Hom}}: cy(s) \longmapsto \lim_{\overleftarrow{t\in \mathbb{I}}} \boldsymbol{hom}\left(cy(s), cy(t)\right)$$

is e −representable, then a **projective e −limit** of the e −regularization $\widetilde{cy}(\mathbb{I})$ in $\boldsymbol{W_e}$ is a representative$(cy(t_0), p_t)_{\boldsymbol{t}\in \mathbb{I}}$of this e −functor, that is a pair $(cy(t_0), p_t)_{\boldsymbol{t}\in \mathbb{I}}$

where $\mathcal{cy}(t_0)$ is an e −manifestation in $\boldsymbol{W_e}$ and $\left(p_t: \mathcal{cy}(t_0) \longmapsto \mathcal{cy}(t)\right)_{\boldsymbol{t} \in \mathbb{I}}$ is a family of e −morphisms such that for any $s, t \in \mathbb{I}$ satisfying $s \leq t$ we have $p_s = h_s^t \circ p_t$.■

A logical and natural question is whether an e − regularization which is projection e −system may be susceptible of a projective e −limit. To give an answer, we will need the following definition.

Definition 9.5 Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ two e −manifestations in $\boldsymbol{W_e} \in \boldsymbol{\mathcal{W}_e}$. Suppose $\boldsymbol{f}$ and $\boldsymbol{g}$ be two e − edges of $\boldsymbol{X}$ into $\boldsymbol{Y}$ and consider the e − functor $\boldsymbol{\mathcal{T}}: \boldsymbol{W_e} \to \boldsymbol{e_{Hom}}: \boldsymbol{Z} \mapsto \boldsymbol{Ker}(\boldsymbol{f}_*, \boldsymbol{g}_*) := \{\boldsymbol{h}: \boldsymbol{h} \in \boldsymbol{hom}(\boldsymbol{Z}, \boldsymbol{X}) \boldsymbol{and} \boldsymbol{f}_*(\boldsymbol{h}) = \boldsymbol{g}_*(\boldsymbol{h})\}$ with

$$\boldsymbol{f}_*: \boldsymbol{hom}(\boldsymbol{Z}, \boldsymbol{X}) \to \boldsymbol{hom}(\boldsymbol{Z}, \boldsymbol{Y}): \boldsymbol{h} \longmapsto \boldsymbol{f}_*(\boldsymbol{h}) = \boldsymbol{f} \circ \boldsymbol{h} \text{ and } \boldsymbol{g}_*: \boldsymbol{hom}(\boldsymbol{Z}, \boldsymbol{X}) \to \boldsymbol{hom}(\boldsymbol{Z}, \boldsymbol{Y}): \boldsymbol{h} \longmapsto \boldsymbol{g}_*(\boldsymbol{h}) = \boldsymbol{g} \circ \boldsymbol{h}.$$

If $\boldsymbol{\mathcal{T}}$ is e −representable, then a representative of this e −functor is said to be a **kernel of double $\boldsymbol{e}$ −edge between $\boldsymbol{X}$ and $\boldsymbol{Y}$**. ■

We have the following result.

Proposition 9.1 Let $\mathcal{cy}: \mathbb{I} \to \boldsymbol{ob}(\boldsymbol{W_e}): t \longmapsto \mathcal{cy}(\boldsymbol{s})$ be a local e—dynamics. Suppose the associated e − regularization $\widetilde{\mathcal{cy}}(\mathbb{I})$ coming from the e − arrangement $\mathcal{cy}(\mathbb{I}) = \left(\mathcal{cy}(\boldsymbol{s})\right)_{\boldsymbol{s} \in \mathbb{I}}$ is a projective e − system on $\boldsymbol{W_e}$ with a kernel of double e − edge between any two e −manifestations $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\boldsymbol{W_e}$. Then the e − regularization $\widetilde{\mathcal{cy}}(\mathbb{I})$ is susceptible of a projective e −limit.

Proof Let us denote by $\left(\mathcal{cy}(\boldsymbol{s}), (h_s^t)\right)_{\boldsymbol{s} \in \mathbb{I}}$ the projective e −system representing the e −regularization. For any pair $(\boldsymbol{s}, \boldsymbol{t})$ such that $\boldsymbol{s} \leq \boldsymbol{t}$, we set $\boldsymbol{Y_{s,t}} = \mathcal{cy}(\boldsymbol{s})$, $\boldsymbol{\alpha_{s,t}} = \boldsymbol{pr_s}$ where $\boldsymbol{pr_s}: \prod_{\boldsymbol{s} \in \mathbb{I}} \mathcal{cy}(\boldsymbol{s}) \to \mathcal{cy}(\boldsymbol{s})$ is the projection at time $\boldsymbol{s}$, and $\boldsymbol{\beta_{s,t}} = h_s^t \circ p_t$. Thus, we have defined two e −morphisms $\boldsymbol{\alpha} = \left(\boldsymbol{\alpha_{s,t}}\right)$ and $\boldsymbol{\beta} = \left(\boldsymbol{\beta_{s,t}}\right)$ of $\prod_{\boldsymbol{s} \in \mathbb{I}} \mathcal{cy}(\boldsymbol{s})$ into $\prod_{\substack{(\boldsymbol{s},\boldsymbol{t}) \in \mathbb{I}^2 \\ \boldsymbol{s} \leq \boldsymbol{t}}} \boldsymbol{Y_{s,t}}$. The kernel of double edge $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is a projective e −limit of the projective e −system

$$\left(\mathcal{cy}(\boldsymbol{s}), (h_s^t)\right)_{\boldsymbol{s} \in \mathbb{I}}$$

representing the e − regularization $\widetilde{\mathcal{cy}}(\mathbb{I})$. The proof is complete. ■

According to this Proposition, **it is important to know if a sub-$\boldsymbol{e}$ − regularization is a projective $\boldsymbol{e}$ − system**, i.e. if possible a description of transformations on the expressions of a sub-e − regularization from one moment to the next one, based on the knowledge of intermediate transformations between the two moments. Indeed, from the previous result, if we know that a sub-e −

regularization is a projection e −system, then this sub-e − regularization will be susceptible of a projective e −limit, which means that it will be possible to determine the single transformation behavior of this sub-e − regularization, throughout the period under examination.

10 Lengths in a cyber-domain

As we shall see just below, the so-called **intrinsic cyber-metric** is a metric possible to define on every cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$. For this metric, the distance between two cyber-elements is the length of the "shortest cyber-track" between these cyber-elements. The term shortest cyber-track will be defined later and is in fact crucial for the understanding of **cyber-geodesics**. To proceed into more details, we will need the following two definitions.

Definition 10.1 Let $\wp$ be a collection of cyber-evolutions with the concatenation operation $*$.

- **i.** A **length** of a cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$ is a (semi-continuous) function $\boldsymbol{\ell}: \wp \longrightarrow [\boldsymbol{0}, +\infty]$satisfying:
  - $\boldsymbol{\ell}(\boldsymbol{cy_2} * \boldsymbol{cy_1}) = \boldsymbol{\ell}(\boldsymbol{cy_2}) + \boldsymbol{\ell}(\boldsymbol{cy_1})$
  - If $\boldsymbol{\mathfrak{U}_X} = \left\{\boldsymbol{Y} \in |\boldsymbol{ob}(\boldsymbol{W_e})|: \boldsymbol{d_{W_e}}(\boldsymbol{X}, \boldsymbol{Y}) < \varepsilon\right\}$ is a neighborhood of $\boldsymbol{X} \in |\boldsymbol{ob}(\boldsymbol{W_e})|$, then
  $$\mathbf{inf}\{\boldsymbol{\ell}(\boldsymbol{cy}): \boldsymbol{cy}(\boldsymbol{0}) = \boldsymbol{X}, \boldsymbol{cy}(\boldsymbol{1}) \in \boldsymbol{W_e} \setminus \boldsymbol{\mathfrak{U}_X}\} > 0.$$
- **ii.** The number $\boldsymbol{\ell}(\boldsymbol{cy})$ is called the **length** of the cyber-evolution $\boldsymbol{cy} \in \wp$.
- **iii.** If the length $\boldsymbol{\ell}(\boldsymbol{cy})$ of a cyber-evolution $\boldsymbol{cy}$ is finite, the cyber-evolution is said to be **rectifiable**.
- **iv.** The completion $|\boldsymbol{ob}(\boldsymbol{W_e})|$ of $\boldsymbol{ob}(\boldsymbol{W_e})$ endowed with the **intrinsic cyber--metric**$\boldsymbol{\varrho_\ell}$defined by
  $$\boldsymbol{\varrho_\ell}(\boldsymbol{X}, \boldsymbol{Y}) \coloneqq \mathbf{inf}\{\boldsymbol{\ell}(\boldsymbol{cy}): \boldsymbol{cy}(\boldsymbol{0}) = \boldsymbol{X}, \boldsymbol{cy}(\boldsymbol{1}) = \boldsymbol{Y}\}$$
  is a metric space called **cyber-length space**.
- **v.** A cyber-length structure $\{|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{\ell}\}$ is **complete**, if for any two cyber-elements $\boldsymbol{X}, \boldsymbol{Y} \in |\boldsymbol{ob}(\boldsymbol{W_e})|$, there is some continuous cyber-evolution$\widetilde{\boldsymbol{cy}} \in \wp$ such that
  $$\widetilde{\boldsymbol{cy}}(\boldsymbol{0}) = \boldsymbol{X},\ \widetilde{\boldsymbol{cy}}(\boldsymbol{1}) = \boldsymbol{Y} \text{ and } \boldsymbol{\varrho_\ell}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{\ell}(\widetilde{\boldsymbol{cy}}).$$

vi. If the underlying cyber-length structure $\{|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{\ell}\}$ is complete, the metric $\boldsymbol{\varrho_\ell}$ is said to be a **complete intrinsic cyber-metric**.

vii. In this last case, the cyber-evolution $\widetilde{\boldsymbol{cy}} \in \wp$ is defined to be the **shortest cyber track** between the cyber-elements $\boldsymbol{X}$ and $\boldsymbol{Y}$.■

Remark 10.1 If $\boldsymbol{S} \equiv \boldsymbol{cy}(\mathbb{I})$ is the cyber-arrangement of a $\boldsymbol{cy}$, then its **length** $\boldsymbol{\ell}(\boldsymbol{S})$ is defined to be the length $\boldsymbol{\ell}(\boldsymbol{cy})$ of $\boldsymbol{cy}$. ■

Remark 10.2 The length $\boldsymbol{\ell}(\boldsymbol{cy})$ of a cyber-evolution $\boldsymbol{cy}$ (or the length $\boldsymbol{\ell}(\boldsymbol{S})$ of the cyber-arrangement $\boldsymbol{S} \equiv \boldsymbol{cy}(\mathbb{I})$ coming from $\boldsymbol{cy}$) is usually defined as

$$\boldsymbol{\ell}(\boldsymbol{cy}) = \boldsymbol{\ell}(\boldsymbol{S}) = \boldsymbol{sup}_{\boldsymbol{0=t_0<t_1<\cdots<t_m=1}\ (\boldsymbol{m}\in\mathbb{N})} \sum_{\boldsymbol{i=0}}^{\boldsymbol{m-1}} \boldsymbol{d}_{\boldsymbol{W_e}}(\boldsymbol{cy}(\boldsymbol{t_i}), \boldsymbol{cy}(\boldsymbol{t_{i+1}})).\ ■$$

Above, we defined the notion of a shortest cyber track between two cyber-elements. In general we will say that **a cyber-evolution $\boldsymbol{cy}$ is a shortest cyber track in $\wp$, if it is a shortest cyber track for all cyber-elements $\boldsymbol{cy}(\boldsymbol{s})$ and $\boldsymbol{cy}(\boldsymbol{t})$ with $\boldsymbol{s}, \boldsymbol{t}$** $\in \mathbb{I} = [\boldsymbol{0}, \boldsymbol{1}]$.

Definition 10.2 **i** A **cyber-geodesic** on a cyber-length space $(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{\varrho_\ell})$ is a cyber-evolution $\boldsymbol{cy}: \mathbb{I} \to \left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d}_{\boldsymbol{W_e}}\right): \boldsymbol{t} \longmapsto cy(\boldsymbol{t})$ in $\wp$ such that for every $\boldsymbol{t} \in \mathbb{I}$, there is a neighborhood $\boldsymbol{U_t} \in \mathbb{I}$ of $\boldsymbol{t}$ such that the restriction $\boldsymbol{cy}/\boldsymbol{U_t}$ of $\boldsymbol{cy}$ onto $\boldsymbol{U_t}$ is a shortest cyber-track.

ii. The cyber-length space $(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{\varrho_\ell})$ is called a **cyber-geodesic space**, if there is the cyber arrangement $\boldsymbol{S} \equiv \boldsymbol{cy}(\mathbb{I})$ of a cyber-evolutionary path $\boldsymbol{cy}$ joining each two cyber-elements $\boldsymbol{X}, \boldsymbol{Y} \in |\boldsymbol{ob}(\boldsymbol{W_e})|$ for which $\boldsymbol{\ell}(\boldsymbol{S}) = \boldsymbol{d}_{\boldsymbol{W_e}}(\boldsymbol{X}, \boldsymbol{Y})$. Such a cyber-arrangement is called a **cyber-geodesic segment** with endpoints $\boldsymbol{X}$ and $\boldsymbol{Y}$.■

There is a simple criterion which assures the existence of cyber-geodesic segments. Since, the cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d}_{\boldsymbol{W_e}}\right)$ is metrically convex, in the sense that for any two cyber-elements $\boldsymbol{P}, \boldsymbol{Q} \in |\boldsymbol{ob}(\boldsymbol{W_e})|$ there is a cyber-element $\boldsymbol{Z} \in |\boldsymbol{ob}(\boldsymbol{W_e})|$, with $\boldsymbol{P} \neq \boldsymbol{Z} \neq \boldsymbol{Q}$, and such that $\boldsymbol{d}_{\boldsymbol{W_e}}(\boldsymbol{P}, \boldsymbol{Z}) + \boldsymbol{d}_{\boldsymbol{W_e}}(\boldsymbol{Z}, \boldsymbol{P}) = \boldsymbol{d}_{\boldsymbol{W_e}}(\boldsymbol{P}, \boldsymbol{Q})$, **Menger's theorem** (see [17]; also [20]) asserts that

Proposition 10.1 Any two cyber-elements of $|\boldsymbol{ob}(\boldsymbol{W_e})|$ are the endpoints of at least one cyber-geodesic segment.■

Every shortest cyber track on a cyber-length space $(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{\varrho_\ell})$ is clearly a cyber-geodesic. However, some cyber-geodesics may fail to be shortest cyber-tracks on large scales. All the same, since each cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$ is a compact, complete metric space, and since, by Proposition 9.1, for any pair of cyber-elements in $|\boldsymbol{ob}(\boldsymbol{W_e})|$ there is a cyber-evolutionary path of finite length joining them, one can exploit a theorem due to Mycielski (see [25]) and obtain the following converse result.

Corollary 10.1 Any pair of two cyber-elements in each cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$ has a shortest cyber track joining them. ■

Further, since any cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$ is complete and, again by Proposition 9.1, it has the property that each two of its cyber-elements can be joined by a rectifiable cyber-evolutionary path, an application of another Mycielski's theorem (see again [25]) guarantees that

Corollary 10.2 Each cyber-length space $(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{\varrho_\ell})$ is complete. ■

Cyber-geodesic spaces provide a fruitful setting for a number of results in metric fixed point theory. There is an interesting general problem of the extent to which these theorems lead to ‘‘approximate’’ fixed point results in cyber-length spaces. For the present, let me give an explicit formula for the cyber-length.

Definition 10.3 The **speed** (**cyber-speed**) of a cyber-evolution $\boldsymbol{cy}: \mathbb{I} \to \left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right): \boldsymbol{t} \longmapsto \boldsymbol{cy}(\boldsymbol{t})$ such at a moment $\boldsymbol{t} \in \mathbb{I} \setminus \{\boldsymbol{1}\}$ is equal to $\boldsymbol{v_{cy}}(\boldsymbol{t}) := \boldsymbol{lim}_{\varepsilon\to 0}\left\{\left[\boldsymbol{d_{W_e}}\left(\boldsymbol{cy}(\boldsymbol{t}), \boldsymbol{cy}(\boldsymbol{t}+\boldsymbol{\varepsilon})\right)\right]/\boldsymbol{\varepsilon}\right\}$ if the limit exists.■

Proposition 10.2 If the speed of a cyber-evolution $\boldsymbol{cy} \in \wp$ exists almost everywhere on $\mathbb{I}$, then its cyber-length is equal to $\boldsymbol{\ell}(\boldsymbol{cy}) = \int_{\boldsymbol{0}}^{\boldsymbol{1}} \boldsymbol{v_{cy}}(\boldsymbol{t})\ \boldsymbol{dt}$. ■

11 Convergence of cyber-evolutions

The notion of uniform convergence is a cornerstone in cyber analysis and will be used repeatedly later on. The difference from pointwise convergence is informally that converging uniformly has to do with how it converges over all of its domain and for pointwise it is sufficient that it converges at every point. Recall that, formally, pointwise convergence is stated as follows.

Definition 11.1 Given a cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$, a sequence of cyber-evolutions $\left(\boldsymbol{cy_k}: \mathbb{I} \to \left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)\right)$ is said to be **pointwise convergent** on $\mathbb{I}$ to the cyber-evolution $\boldsymbol{cy}: \mathbb{I} \to \left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$ if

for every $\boldsymbol{t} \in \mathbb{I}$, there exists, for every $\boldsymbol{\varepsilon} > \boldsymbol{0}$, an $\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{t}) \in \mathbb{N}$ such that for every $\boldsymbol{k} \geq \boldsymbol{K}$,

$$\boldsymbol{d_{W_e}}\left(\boldsymbol{cy_k}(\boldsymbol{t}) - \boldsymbol{cy}(\boldsymbol{t})\right) < \varepsilon.$$

This will further on be denoted as $\boldsymbol{lim_{k\to\infty} cy_k}(\boldsymbol{t}) = \boldsymbol{cy}(\boldsymbol{t})$.■

On the other hand, uniform convergence is stated as:

Definition 11.2 Given a cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$, a sequence of cyber-evolutions

$$\left(\boldsymbol{cy_k}: \mathbb{I} \to \left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)\right)$$

is said to be **uniformly convergent** on $\mathbb{I}$ to the cyber-evolution $\boldsymbol{cy}: \mathbb{I} \to \left(|\boldsymbol{ob}(\boldsymbol{W_e})|, \boldsymbol{d_{W_e}}\right)$ if

for every $\boldsymbol{\varepsilon} > \boldsymbol{0}$, an $\boldsymbol{K} \in \mathbb{N}$ such that for every $\boldsymbol{t} \in \mathbb{I}$ and $\boldsymbol{k} \geq \boldsymbol{K}$,

$$\boldsymbol{d_{W_e}}\left(\boldsymbol{cy_k}(\boldsymbol{t}) - \boldsymbol{cy}(\boldsymbol{t})\right) < \varepsilon.■$$

By Remark 6.2, if $\boldsymbol{W_e} = \boldsymbol{\mathcal{E}_{\mathcal{C}}}$, the topology of the locally finite e −graph $\mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}$ induced by the metric $\boldsymbol{d}_{\mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}}$ coincides with Diestel's metrizable topology ([8]). So, the topology of its unique e −graph completion $\left|\mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}\right|$ induced by the same metric $\boldsymbol{d}_{\mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}}$ is a compact topology. Since $\boldsymbol{d}_{\mathfrak{G}_{\mathcal{E}_{\mathcal{C}}}}$ can be identified with the metric $\boldsymbol{d}_{\mathcal{E}_{\mathcal{C}}}$ in the set of objects $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$ of the e −category $\boldsymbol{\mathcal{E}_{\mathcal{C}}}$, we infer that the unique completion $|\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})|$ of $\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})$ is a **compact** space with respect to the metric $\boldsymbol{d}_{\mathcal{E}_{\mathcal{C}}}$. A direct application of Arzela-Ascoli theorem (see, for instance, [3]), proves the following result.

Proposition 11.1 Any sequence of cyber-evolutions with uniformly bounded lengths on the cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})|, \boldsymbol{d}_{\mathcal{E}_{\mathcal{C}}}\right)$ has a uniformly converging subsequence.■

Since, by Proposition 9.1, the cyber-domain $\left(|\boldsymbol{ob}(\boldsymbol{\mathcal{E}_{\mathcal{C}}})|, \boldsymbol{d}_{\mathcal{E}_{\mathcal{C}}}\right)$ has the property that each two of its cyber-elements can be joined by a rectifiable cyber-evolution path, we infer, from Proposition 10.1, that

Corollary 11.1 There exists a shortest cyber-track between any two cyber-elements $\boldsymbol{X}$ and $\boldsymbol{Y}$. ■

Further, using the semi-continuity of cyber-length, we show the next result.

Proposition 11.2 If a sequence $(\boldsymbol{c\mathcal{Y}_k})$ of shortest cyber-tracks on the cyber-domain $\left(|\boldsymbol{ob}(\mathcal{E}_{\mathcal{C}})|, \boldsymbol{d}_{\mathcal{E}_{\mathcal{C}}}\right)$ converges uniformly to a cyber-evolution $\boldsymbol{c\mathcal{Y}}$, then its limit $\boldsymbol{c\mathcal{Y}}$ is also a shortest cyber-track. ■

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