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Global random attractor for the strongly damped stochastic wave equation with white noise and nonlinear term

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Abstract

In this paper, we study the long time behavior of solution to the stochastic strongly damped wave equation with white noise. We first prove the wellness of the solutions, then we establish the existence of global random attractor.

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1 Introduction

In this paper, we consider the large-time behavior of the following initial

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boundary value problem for the stochastic strongly damped wave equation with white noise in a bounded domain $\mathcal{D} \subset \mathrm{R}$ with smooth boundary

$$u_{tt} - \Delta u - \alpha\Delta u_t + \beta u_t + f(u) - g(x) = q\dot{W}, (x,t) \in \mathcal{D} \times [0,+\infty), \quad (1)$$

$$u(x,0) = u_0(x);\ u_t(x,0) = u_1(x), \quad x \in \mathcal{D}, \quad (2)$$

$$u(x,t)|_{\partial\Omega} = 0, \qquad (x,t) \in \partial\mathcal{D} \times [0,+\infty), \quad (3)$$

where $(u_0, u_1) \in H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$, and α, β are positive constants, $u = u(x,t)$ is a real-valued function on $\mathcal{D} \times [0,+\infty)$. $\dot{W}$ is a scalar Gaussian white noise, that is, $W(t)$ is a two-sided wiener process.

The functions $f : \mathrm{R} \to \mathrm{R}$ and $g, q : \mathcal{D} \to \mathrm{R}$ satisfies the following assumptions:

(i) $g \in H_0^1(\mathcal{D})$, while $q \in H^2(\mathcal{D}) \times H_0^1(\mathcal{D})$ is not identically equal to zero.

(ii) The nonlinear term f satisfies

$$|f'(u)| \le C_0,\ |f(u)| \le C_1, \quad \forall u \in \mathrm{R}; \quad (4)$$

$$|f'(u) - f'(v)| \le C_2|u - v|, \quad \forall u, v \in \mathrm{R}, \quad (5)$$

where C_0, C_1, C_2 are positive constants.

It's well known that the long time behavior of many dynamical system generated by evolution equations can be described naturally in term of attractors of corresponding semigroups. Global attractor is a basic concept in the study of the asymptotic behavior of solutions for the nonlinear evolution equations with various dissipation . There have been many researches on the long-time behavior of solutions to the nonlinear damped wave equations, While the random attractor is an analogue of global attractors for deterministic dynamical systems. Since the foundational work in [9,10], the existence of random attractors has been investigated by many authors, see,e.g., [1-8,11-17]. In this work, we apply the means in [4] to provide the existence of a random attractor, for the random dynamical system generated by the initial value problem (1.1)–(1.3). The key is to deal with the nonlinear terms and strongly damped term $-\Delta u_t$, $-\Delta u_t$ is difficult to be handled. so we aimed at show that it's dissipative and the solution is bounded and continuous with respect to initial value. Hence we can discover the random attractor, which has finite fractal and Hausdorff dimension. In this paper, we use the method introduced in [3], so that we needn't divide the equation into two parts.

The rest of this paper is organized as follows. In section 2, we introduce basic concepts concerning random attractor. In section 3, we obtain the existence of the uniqueness random attractor,which has fractal and Hausdorff dimension.

2 Preliminary Notes

Let $(X, \|\cdot\|)$ is a separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\Omega = \{w \in C(\mathrm{R}, \mathrm{R});\ w(0) = 0\}$ is endowed with compact-open topology, $\mathbb{P}$ is the corresponding Wiener measure, and $\mathcal{F}$ is the $\mathbb{P}$-completion of Borel $\sigma - algebra$ on Ω. Let $\{\theta_t : \Omega \to \Omega, t \in \mathrm{R}\}$ be a family of measure preserving transformations such that $(t, w) \mapsto \theta_t w$ is measurable, $\theta_0 = Id_\Omega, \theta_{t+s} = \theta_t\theta_s$, for all $s, t \in \mathrm{R}$. The space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathrm{R}})$ is called the metric dynamical system on the probability space$(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1. ([6]) *A continuous random dynamical system on* X *over* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathrm{R}})$ *is a* $(\mathcal{B}(R^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$*-measurable mapping*

$$\varphi : R^+ \times \Omega \times X \to X, \quad (t, w, u) \mapsto \varphi(t, w, u)$$

such that the following properties hold

(1) $\varphi(0, w, u) = u$ *for all* $w \in \Omega$ *and* $u \in X$*;*

(2) $\varphi(t + s, w, \cdot) = \varphi(t, \theta_s w, \varphi(s, w, \cdot))$ *for all* $s, t \geq 0$*;*

(3) $\varphi(t, w, \cdot) : X \to X$ *is continuous for all* $t \geq 0$*.*

Definition 2.2. ([6])

(1) *A set-valued mapping* $\{D(w) : \Omega \to 2^X, w \mapsto D(w)\}$*, is said to be a random set if the mapping* $w \mapsto d(u, D(w))$ *is measurable for any* $u \in X$*. If* $D(w)$ *is also closed(compact) for each* $w \in \Omega$*,* $\{D(w)\}$ *is called a random closed (compact) set. A random set* $\{D(w)\}$ *is said to be bounded if there exists* $u_0 \in X$ *and a random variable* $R(w) > 0$ *such that*

$$D(w) \subset \{u \in X, : \|u - u_0\|_X \leq R(w)\} \ for\ all\ \ w > 0.$$

(2) *A random set* $\{D(w)\}$ *is called tempered provided for* $\mathbb{P}-a.e.w \in \Omega$,

$$\lim_{t\to+\infty} e^{-\beta t} d(D(\theta_{-t}w)) = 0 \quad for\ all\ \beta > 0,$$

where $d(D) = \sup\{\|b\|_X : b \in D\}$.

Let $\mathcal{D}$ *be the set of all random tempered sets in* X.

(3) *A random set* $\{B(w)\}$ *is said to be a random absorbing set if for any tempered random set* $\{D(w)\}$*, and* $\mathbb{P}-a.e.w \in \Omega$*, there exists* $t_0(w)$ *such that*

$$\varphi(t, \theta_{-t}w, D(\theta_{-t}w)) \subset B(w) \quad for\ all\ t \geq t_0(w).$$

(4) *A random set* $\{B_1(w)\}$ *is said to be a random attracting set if for any tempered random set* $\{D(w)\}$*,,and* $\mathbb{P}-a.e.w \in \Omega$*,we have*

$$\lim_{t\to+\infty} d_H(\varphi(t, \theta_{-t}w, D(\theta_{-t}w), B_1(w)) = 0,$$

where d_H *is the Hausdorff semi-distance given by* $d_H(E, F) = \sup_u \in E \inf_v \in F\|u - v\|_X$ *for all* $E, F \subset X$.

(5) φ *is said to be asymptotically compact in* X *if for* $\mathbb{P}-a.e.w \in \Omega$, $\{\varphi(t_n, \theta_{-t_n}w, x_n)\}_{n=1}^{\infty}$ *has a convergent subsequence in* X *whenever* $t_n \to +\infty$*, and* $x_n \in B(\theta_{-t_n}w)$ *with* $B(w) \in \mathcal{D}$.

(6) *A random compact set* $\{A(w)\}$ *is said to be a random attractor if it is a random attracting set and* $\varphi(t, w, A(w)) = A(\theta_t w)$ *for* $\mathbb{P}-a.e.w \in \Omega$ *and all* $t \geq 0$.

Theorem 2.3. *Let* φ *be a continuous random dynamical system with state space* X *over* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathrm{R}})$*. If there is a closed random absorbing set* $\{B(w)\}$ *of* φ *and* φ *is asymptotically compact in* X*, then* $\{A(w)\}$ *is a random attractor of* φ *,where*

$$A(w) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}w, B(\theta_{-\tau}w))}, \quad w \in \Omega.$$

Moreover, $\{A(w)\}$ *is the unique random attractor of* φ.

Definition 2.4. [5] *Let M be metric space and A be a bounded subset of M. The measure of noncompactness $\gamma(A)$ of A is defined by*

$$\gamma(A) = inf\{\delta > 0 | A \; admits \; finite \; cover \; by \; sets \; whose \; diameter \; \leq \delta\}.$$

Definition 2.5. *A C^0 semigroup $\{S(t)\}_{t\geq 0}$ in a complete metric space M is called $w-limit$ compact if for every bounded subset B of M, and for every $\varepsilon > 0$, there is a $t(B) > 0$, such that*

$$\gamma(\bigcup_{t\geq t(B)} S(t)B) \leq \varepsilon.$$

Definition 2.6. *$Condition(C)$[5] For any bounded set B of a Banach space X and for any $\varepsilon > 0$, there exists a $t(B) > 0$ and a finite dimensional subspace X_1 of X such that $\{P_m S(t)B\}$ is bounded and*

$$\|(I - P_m)S(t)x\| < \varepsilon, \;\; for \; t \geq t(B), \; x \in B,$$

where I is the identity and $P_m : X \to X_1$ is a bounded projector.

Theorem 2.7. [3] *Let $\{S(t)\}_{t\geq 0}$ be a C^0 semigroup in a complete metric space M. Then $\{S(t)\}_{t\geq 0}$ has a group attractor $\mathcal{A}$ in M if and only if*

(1) *$\{S(t)\}_{t\geq 0}$ is $w-limit$ compact, and*

(2) *there is a bounded absorbing set $B \subset M$.*

Lemma 2.8. [3] *Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a C^0 semigroup in X.*

(1) *If $Condition(C)$ holds, then $\{S(t)\}_{t\geq 0}$ is $w-limit$ compact.*

(2) *Let X be uniformly convex Banach space. Then $\{S(t)\}_{t\geq 0}$ is $w-limit$ compact if and only if $Condition(C)$ holds.*

Theorem 2.9. [3] *Let X be a uniformly convex Banach space(especially a Hilbert space). Then the C^0 semigroup $\{S(t)\}_{t\geq 0}$ has a group attractor if and only if*

(1) *there is a bounded absorbing set $B \subset M$.*

(2) *condition (C) holds.*

3 The random attractor

In this section, our objection is to show that the well-posed and the existence of global attractor for the initial boundary value problem (1.1)–(1.3).

Let $\frac{1}{\alpha} < \varepsilon < \min\{\frac{1}{\alpha}, \beta\}$ and $\alpha > 1$, then by the transformation $v(x,t) = u_t + \varepsilon u - qW$, the initial boundary value problem (1.1)–(1.3) is equivalent to

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = v - \varepsilon u + qW, \\ \\ \frac{\mathrm{d}v}{\mathrm{d}t} = (1-\alpha\varepsilon)\Delta u + \alpha\Delta v + \varepsilon(\beta-\varepsilon)u + (\varepsilon-\beta)v + \\ \qquad\qquad +(\varepsilon-\beta)qW + \alpha\Delta qW - f(u) + g \end{cases} \tag{1}$$

with the initial value conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \tag{2}$$

where $v_0(x) = u_1 + \varepsilon u_0$.

Let $E = H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$,which is endowed with the usual norm

$$\|Y\|^2_{H_0^1\times L^2} = \|\nabla u\|^2 + \|u\|^2\| + \|v\|^2, \ for\ Y = (u,v)^\top \in E, \tag{3}$$

where $\|\cdot\|$ denote the norm in $L^2(\mathcal{D})$ and $\top$ stands for the transposition.

For convenient, we now define a new norm $\|\cdot\|_E$ by

$$\|Y\|^2_E = (1-\alpha\varepsilon)\|\nabla u\|^2 + \varepsilon(\beta-\varepsilon)\|u\|^2\| + \|v\|^2, \ for\ Y = (u,v)^\top \in E, \tag{4}$$

it is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H_0^1\times L^2}$ in (3.3).

Theorem 3.1. *Assume that (i) and (ii) hold for all* $(u_0, v_0)^\top \in E$*,* α, β *are the positive constants. Then the initial boundary value problem (3.1) has unique solution* $(u, v)^\top \in E$*, which is continuous with respect to* $(u_0, v_0)^\top \in E$ *for all* $t > 0$*.*

Proof. Taking the inner product of the second equation of (3.1) with v in $L^2(\mathcal{D})$, we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 &= (1-\alpha\varepsilon)(\Delta u, v) - \alpha\|\nabla v\|^2 + \varepsilon(\beta-\varepsilon)(u,v) \\ &\quad + (\varepsilon-\beta)\|v\|^2 + ((\varepsilon-\beta)qW + \alpha\Delta qW, v) - (f(u), v) + (g, v). \quad (5)\end{aligned}$$

Since $v = u_t + \varepsilon u - qW$, we deal with the terms in (3.5) one by one as follows

$$\begin{aligned}(1-\alpha\varepsilon)(\Delta u, v) &= -\frac{1-\alpha\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 - \varepsilon(1-\alpha\varepsilon)|\nabla u\|^2 \\ +(1-\alpha\epsilon)(u, \Delta qW) &\geq -\frac{1-\alpha\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 - \varepsilon(1-\alpha\varepsilon)|\nabla u\|^2 - \\ &\quad \frac{\varepsilon(\beta-\varepsilon)(1-\alpha\varepsilon)}{2}\|u\|^2 - \frac{(1-\alpha\varepsilon)}{2\varepsilon(\beta-\varepsilon)}\|\Delta q\|^2\|W\|^2 \quad (6)\end{aligned}$$

$$\begin{aligned}\varepsilon(\beta-\varepsilon)(u,v) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(\beta-\varepsilon)\|u\|^2 + \varepsilon^2(\beta-\varepsilon)\|u\|^2 + \varepsilon(\beta-\varepsilon)(u, qW) \\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(\beta-\varepsilon)\|u\|^2 + \varepsilon^2(\beta-\varepsilon)\|u\|^2 - \frac{\epsilon(\beta-\varepsilon)(1-\alpha\varepsilon)}{2}\|u\|^2 \\ &\quad -\frac{\epsilon(\beta-\varepsilon)}{2(1-\alpha\varepsilon)}\|q\|^2\|W\|^2 \quad (7)\end{aligned}$$

$$(\varepsilon-\beta)(qW, v) \leq \frac{\alpha\lambda_1+\beta}{4}\|v\|^2 + \frac{(\beta-\varepsilon)^2}{\lambda_1^2(\alpha\lambda_1+\beta)}\|\Delta q\|^2\|W\|^2; \quad (8)$$

$$\alpha(\Delta qW, v) \leq \frac{\alpha\lambda_1+\beta}{4}\|v\|^2 + \frac{\alpha^2}{\alpha\lambda_1+\beta}\|\Delta q\|^2\|W\|^2; \quad (9)$$

$$(-f(u), v) \leq C_1\|v\|^2 \leq \frac{\alpha\lambda_1+\beta}{4}\|v\|^2 + \frac{C_1^2}{\alpha\lambda_1+\beta}; \quad (10)$$

$$(g, v) \leq \|g\|\|v\| \leq \frac{\alpha\lambda_1+\beta}{4}\|v\|^2 + \frac{\|g\|^2}{\alpha\lambda_1+\beta}; \quad (11)$$

$$-\alpha\|\nabla v\|^2 \leq -\alpha\lambda_1\|v\|^2, \quad (12)$$

where $\lambda_1$is the first eigenvalue of $-\Delta$ in $H_0^1(\mathcal{D})$.

By (3.6)–(3.12), it follows from that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2+(1-\alpha\varepsilon)\|\nabla u\|^2+\varepsilon(\beta-\varepsilon)\|u\|^2)\\
&\le -\varepsilon(\|v\|^2+(1-\alpha\varepsilon)\|\nabla u\|^2+\varepsilon(\beta-\varepsilon)\|u\|^2)+\varepsilon(\beta-\varepsilon)(1-\alpha\varepsilon)\|u\|^2\\
&+2\varepsilon\|v\|^2+k_\varepsilon\|\Delta q\|^2\|W\|^2+\frac{C_1^2+\|g\|^2}{\alpha\lambda_1+\beta}\\
&\le -(\alpha\varepsilon-1-\varepsilon)[\|v\|^2+(1-\alpha\varepsilon)\|\nabla u\|^2+\varepsilon(\beta-\varepsilon)\|u\|^2]\\
&+k_\varepsilon\|\Delta q\|^2\|W\|^2+\frac{C_1^2+\|g\|^2}{\alpha\lambda_1+\beta}.
\end{aligned}
\tag{13}
$$

where $k_\varepsilon=\frac{1-\alpha\varepsilon}{2\varepsilon(\beta-\varepsilon)}+\frac{\varepsilon(\beta-\varepsilon)}{2(1-\alpha\varepsilon)\lambda_1^2}+\frac{(\beta-\varepsilon)^2}{\lambda_1^2(\alpha\lambda_1+\beta)}+\frac{\alpha^2}{\alpha\lambda_1+\beta}$.

Setting $\varphi=(u,v)\in E$, recalling the new norm$\|\cdot\|_E$ in (3.4), we obtain from (3.13) and (3.4) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|Y\|_E^2\le -2(\alpha\varepsilon-1-\varepsilon)\|\varphi\|_E^2+C_\varepsilon(\|\Delta q\|^2\|W\|^2+\|g\|^2+C_1^2). \tag{14}
$$

Using the Gronwall lemma then gives

$$
\|Y\|_E^2\le e^{-2(\alpha\varepsilon-1-\varepsilon)t}\|\varphi_0\|_E^2+\frac{C_\varepsilon}{2(\alpha\varepsilon-1-\varepsilon)}(\|\Delta q\|^2\|W\|^2+\|g\|^2+C_1^2). \tag{15}
$$

Substituting w by θ_{-t} and taking W by w, then we have from (3.15) that

$$
\begin{aligned}
\|\varphi(t,\theta_{-t}w,\varphi_0(\theta_{-t}w))\|_E^2 &\le e^{-2(\alpha\varepsilon-1-\varepsilon)t}\|\varphi_0(\theta_{-t}w)\|_E^2\\
&+\frac{C_\varepsilon}{2(\alpha\varepsilon-1-\varepsilon)}(\|\Delta q\|^2\|W\|^2+\|g\|^2+C_1^2).
\end{aligned}
\tag{16}
$$

(3.16) implies that the existence of the solution of random wave equation (3.1), it's easy to check that the solution is unique. so, the proof is completed. □

By the theorem 3.1,we obtain the global smooth solution (u,u_t) continuously depends on the initial value (u_0,u_1), the initial boundary value problem (1.1)–(1.3) generates a continuous semigroup $\{S(t)\}_{t\ge 0}, S(t):X\to X$; $(u,u_t)=S(t)(u_0,u_1)$. Choosing

$$
R=\frac{C_\varepsilon}{2(\alpha\varepsilon-1-\varepsilon)}(\|\Delta q\|^2\|W\|^2+\|g\|^2+C_1^2),
$$

then $B_R=\{(u,u_t)|\ \|(u,u_t)\|_X\le R\}$ is a bounded absorbing set for the semigroup $\{S(t)\}_{t\ge 0}$ generated by (1.1)–(1.3).

Under the assumption $(i),(ii)$, we can get the nonlinear term $g(u)$ is compact and continuous, $f(x)$ is continuous. Next, our object is to show that the C^0 semigroup $\{S(t)\}_{t\ge 0}$ satisfies *cinditionC*.

Theorem 3.2. *Assume that* (*i*) *and* (*ii*) *hold for all* $(u_0, v_0) \in E$,α, β *are positive constants. Then the* C^0 *semigroup* $\{S(t)\}_{t\geq 0}$ *associated with initial value problem* (3.1) *satisfies cinditionC, that is, there exists* $m \in \mathrm{N}$ *and* $T = T(B, R)$, *for any* $N \geq m, t \geq T$ *such that*

$$\|(I - P_m)\varphi(t, \theta_{-t}w, \varphi_0(\theta_{-t}w))\|_E^2 \leq \frac{k_{1\varepsilon}\|\Delta q_2\|^2\|W\|^2 + \varepsilon}{2(\alpha\varepsilon - 1 - \varepsilon)}$$

Proof. Let λ_j be the eigenvalues of $-\Delta u$ and w_j be the corresponding eigenvectors, $j = 1, 2, ...$, without loss of generality, we can assume that $\lambda_1 < \lambda_2 < ...$, and $\lim\limits_{m\to\infty} \lambda_m = \infty$.

It is well known that $\{w_j\}_{j=1}^{\infty}$ form an orthogonal basis of H_0^1. We write

$$H_m = span\{w_1, w_2, ..., w_m\}$$

Since $f \in H_0^1$ and f is compact, for any $\varepsilon > 0$, there exists some $m \in \mathbf{N}$ such that

$$\|(I - P_m)f\| \leq \frac{\varepsilon}{2}, \tag{17}$$

$$\|(I - P_m)g\| \leq \frac{\varepsilon}{2}, \quad for\ all\ u \in B_R(0, R) \tag{18}$$

where $P_m : H_0^1 \to H_m$ is orthogonal projection and R is the radius of the absorbing set. For any $(u, u_t) \in E$, we write

$$(u, u_t) = (P_m u, P_m u_t) + ((I - P_m)u, (I - P_m)u_t) = (u_1, u_{1t}) + (u_2, u_{2t}). \tag{19}$$

We note that

$$q_2 = (I - P_m)q, \quad g_2 = (I - P_m)g, \quad f_2 = (I - P_m)f,$$

Taking the inner product of the second equation of (3.1) with v_2 in $L^2(\mathcal{D})$, After a computation like in the proof of Theorem 3.1, we can yield that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_2\|^2 = &\ (1 - \alpha\varepsilon)(\Delta u_2, v_2) - \alpha\|\nabla v_2\|^2 + \varepsilon(\beta - \varepsilon)(u_2, v_2) \\ &+ (\varepsilon - \beta)\|v_2\|^2 + ((\varepsilon - \beta)q_2 W + \alpha\Delta q_2 W, v_2) \\ &- (f_2(u), v_2) + (g_2, v_2).\end{aligned} \tag{20}$$

This is the same as in the proof of the Theorem 3.1, except for a replacement of λ_1 with λ_{m+1} and a choice of $k_{1\varepsilon}$. Combined with (3.17) and (3.18), then we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v_2\|^2+(1-\alpha\varepsilon)\|\nabla u_2\|^2+\varepsilon(\beta-\varepsilon)\|u_2\|^2)\\&\leq -(\alpha\varepsilon-1-\varepsilon)[\|v_2\|^2+(1-\alpha\varepsilon)\|\nabla u_2\|^2+\varepsilon(\beta-\varepsilon)\|u_2\|^2]\\&+k_{1\varepsilon}\|\Delta q_2\|^2\|W\|^2+\varepsilon.\end{aligned} \tag{21}$$

By Gronwall lamma , we can get

$$\begin{aligned}&(1-\alpha\varepsilon)\|\nabla u_2\|^2+\varepsilon(\beta-\varepsilon)\|u_2\|^2\|+\|v_2\|^2\\ \leq\ & e^{-2(\alpha\varepsilon-1-\varepsilon)t}\|\varphi_0(\theta_{-t}w)\|_E^2+\frac{k_{1\varepsilon}\|\Delta q_2\|^2\|W\|^2+\varepsilon}{2(\alpha\varepsilon-1-\varepsilon)},\end{aligned} \tag{22}$$

i.e.

$$(1-\alpha\varepsilon)\|\nabla u_2\|^2+\varepsilon(\beta-\varepsilon)\|u_2\|^2\|+\|v_2\|^2\leq\frac{k_{1\varepsilon}\|\Delta q_2\|^2\|W\|^2+\varepsilon}{2(\alpha\varepsilon-1-\varepsilon)}. \tag{23}$$

Substituting w by θ_{-t} and taking W by w, then we have from (3.15) that

$$\|(I-P_m)\varphi(t,\theta_{-t}w,\varphi_0(\theta_{-t}w))\|_E^2\leq\frac{k_{1\varepsilon}\|\Delta q_2\|^2\|W\|^2+\varepsilon}{2(\alpha\varepsilon-1-\varepsilon)}$$

this shows that $Condition(C)$ is satisfied, and the proof is completed. □

Theorem 3.3. *Assume that* (*i*) *and* (*ii*) *hold for all* $(u_0,v_0)\in E$,α,β *are the positive constants. Then the* C^0 *semigroup* $\{S(t)\}_{t\geq 0}$ *associated with initial value problem* (3.1) *has a unique global attractor in E.*

Lemma 3.4. *Let* X *be a Banach space,* $S:X\to X$ *be a continuous map. If* S *is bounded, that is, takes bounded sets to bounded sets, and* $S=P+U$*, where* P *is globally Lipschitz continuous with a Lipschitz constant* $k\in(0,1)/$*,and* U *is conditionally completely continuous, then* S *is a* $\beta-$*contraction, and hence, an* $\alpha-$*contraction (see [8] for the definition of both* $\beta-$*contraction and* $\alpha-$*contraction).*

Theorem 3.5. *Assume that* (*i*) *and* (*ii*) *hold for all* $(u_0,v_0)\in E$,α,β *are the positive constants. Then the unique global attractor in* E *for the* C^0 *semigroup* $\{S(t)\}_{t\geq 0}$ *associated with initial value problem* (3.1) *is connected and has finite fractal and Hausdorff dimension..*

The global estimate (3.15) shows that the continuous semigroup $S(t)$ is bounded for $t \geq 0$. Since P_m is completely continuous as $t > 0$, by Lemma 4.1, $S(t)$ is an $\alpha-$ contraction as $t > T_0$. On the other hand , by the estimation in (3.23), we can get $S(t)$ is an $\alpha-$ contraction as $t > T_0$. And as a direct consequence of Theorem 2.8.1 of Hale [8], A has finite fractal and Hausdorff dimension.

References

[1] J.C. Robinson, *Infinite dimensional dynamical systems*, Cambridge University Press, London, 2001.

[2] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1988.

[3] C.K. Zhong, M.H.Yang and C.Y.Sun, The existence of global attractors for the norm-to-weak continuous semigroup, *J. Differential Equations*, **223**, (2006).

[4] Y.Q. Xie and C.K. Zhong, The existence of global attractor for a class of nonlinear evolution equation, *J. Math. Anan. Appl.*, **336**, (2007), 54-69.

[5] V. Pata and M. Squassina, On the strongly damped wave equation, *Comm. Math. Phys.*, **253**, (2005), 511-533.

[6] I. Chueshov, *Monotone random system theory and applications*, Springer-Verlag, 2002.

[7] C.Y. Sun, S.h. Wang and C.K. Zhong, Global attractors for a nonlassical diffusion equation, *Acta Math.Sin.*, (Engl.Ser.) **26B**(3), (2005), 1-8.

[8] J. Hale, Asymptotic Behavior of Dissipative Systems, *American Mathematical Society*, Providence, RI, 1988.

[9] Y.Q. Xi and C.K. Zhong, Asymptotic behavior of a class of nonlinear evolution equation, *Nonlinear Anal.*, **71**, (2009), 509-515.

[10] M.H. Yang, J.Q. Duan and Peter Kloeden, Asymptotic behavior of solutions for random wave equations with nonliear damping and white noise, *Nonlinear Anal: Real World applications*, **12**, (2011), 464-478.

[11] Z.J. Wang and S.F. Zhou, Random attractor of the stochastic strongly damped wave equation, *Commun Nonlinear Sci Numer Simulat.*, **17**, (2012), 1649-1658.

[12] Z.J. Wang and S.F. Zhou, Random attractor of the stochastic strongly damped wave equation with multiplicative noise on unbounded domain, *Nonlinear Anal: Real World applications*, **12**, (2011), 3468-3482.

[13] S.F. Zhou and L. Wei, A random attractor for a stochastic second order lattice system with random coupled coefficients, *J. Math. Anal. Appl.*, **395**, (2012), 42-55.

[14] Z.J. Yang, Global attractor for a nonlinear wave equation arising in elastic waveguide model, *Nonlinear Anal.*, **70**, (2009), 2132-2142.

[15] Z.J. Yan and X. Li, Finite-dimensional attractors for the Kirchhoff equation with a strong dissipation, *J. Math. Anal. Appl.*, **375**, (2011), 579-593.

[16] Y.Q. Xie and C.K. Zhong, Asymptotic behavior of a class of nonlinear evolution equations, *Nonlinear Anal.*, **71**, (2009), 5095-5105.

[17] S.B. Wang and G.W. Chen, Cauchy problem of the generalized double dispersion equation, *Nonlinear Anal.*, **64**, (2006), 159-173.