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Mixed Type Second-order Duality for a Nondifferentiable Continuous Programming Problem

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Abstract

A mixed type second-order dual is formulated for a class of continuous programming problems in which the integrand of the objective functional contains square root of positive semi-definite quadratic form; hence it is nondifferentiable. Under second-order pseudoinvexity and second-order quasi-invexity, various duality theorems are proved for this pair of dual nondifferentiable continuous programming problems. A pair of dual continuous programming problems with natural boundary values is constructed and it is briefly indicated that the duality results for this pair of problems can be validated analogously to those for the earlier models. Lastly, it is pointed out that our duality results can be regarded as dynamic generalizations of those for a nondifferentiable nonlinear programming problem, already treated in the literature.

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1 Introduction

A number of researchers have studied second-order duality in mathematical programming. A second order dual to a Nonlinear programming problem was first formulated by Mangasarian [1] .Subsequently Mond [2] established various duality theorem under a condition which is called second-order convexity, which is much simpler than that used by Mangasarian [1]. Mond and weir [3] reformulated the second-order and higher-order duals to validate duality results. It is found that second-order dual to a mathematical programming problem offers a tighter bound and hence enjoys computational advantage over a first-order dual.

In the spirit of Mangasarian [1], Chen [4] formulated second-order dual for a constrained variational problem and established various duality results under an involved invexity-like assumption. Later, Husain et al. [5], studied Mond-Weir type second-order duality for the problem of [4], by introducing continuous-time version of second-order invexity and generalized second-order invexity, validated usual duality results. Subsequently Husain and Masoodi [6] presented Wolf type duality while Husain and Srivastava [7] formulated Mond-weir type dual for a class of continuous programming containing square root of a quadratic form to relax to the assumption of second-order pseudoinvexity and second-order quasi-invexity.

In this paper, in order to combine dual formulations of [6] and [7], a mix type dual to the non-differentiable continuous programming problem of Chandra et al. [8] is constructed and a number of duality results are proved under appropriate generalized second-order invexity established. A relation between our duality results and those of a nondifferentiable nonlinear programming problem is pointed out through natural boundary value variational problems.

2 Related Pre-requisites

Let I = [a, b] be a real interval, $\phi : I \times R^n \times R^n \to R$ and $\psi : I \times R^n \times R^n \to R^m$ be twice continuously differentiable functions. In order to consider $\phi(t, x(t), \dot{x}(t))$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, denoted by ϕ_x *and* $\phi_{\dot{x}}$, the first order of ϕ with respect to $x(t)$ *and* $\dot{x}(t)$, respectively, that is,

$$\phi_x = \left(\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \cdots, \frac{\partial \phi}{\partial x^n} \right)^T, \quad \phi_{\dot{x}} = \left(\frac{\partial \phi}{\partial \dot{x}^1}, \frac{\partial \phi}{\partial \dot{x}^2}, \cdots, \frac{\partial \phi}{\partial \dot{x}^n} \right)^T.$$

Denote by ϕ_{xx} the Hessian matrix of ϕ and ψ_x the $m \times n$ Jacobian matrices respectively, that is $\phi_{xx} = \left(\frac{\partial^2 \phi}{\partial x^i \partial x^j} \right)$, $i, j = 1, 2, \ldots n$, ψ_x the $m \times n$ Jacobian matrix

$$\psi_x = \begin{pmatrix} \frac{\partial \psi^1}{\partial x^1} & \frac{\partial \psi^1}{\partial x^2} \cdots & \frac{\partial \psi^1}{\partial x^n} \\ \frac{\partial \psi^2}{\partial x^1} & \frac{\partial \psi^2}{\partial x^2} \cdots & \frac{\partial \psi^2}{\partial x^n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \psi^m}{\partial x^1} & \frac{\partial \psi^m}{\partial x^2} \cdots & \frac{\partial \psi^m}{\partial x^n} \end{pmatrix}_{m \times n}.$$

The symbols $\phi_{\dot{x}}$, $\phi_{\dot{x}x}$, $\phi_{x\dot{x}}$ *and* $\psi_{\dot{x}}$ have analogous representations.

Designate by X the space of piecewise smooth functions $x : I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = D\,x \Leftrightarrow x(t) = \int_a^t u(s)\,ds,$$

Thus $\frac{d}{dt} = D$ except at discontinuities.

Now we incorporate the following definitions which are required in the subsequent analysis.

Definition2.1 (Second-order Invex) If there exist a vector function $\eta = \eta(t, x, \bar{x}) \in R^n$ where $\eta : I \times R^n \times R^n \to R^n$ and with $\eta = 0$ at t = *a* and t = *b*, such that for a scalar function $\phi(t, x, \dot{x})$, the functional $\int_I \phi(t, x, \dot{x})\,dt$ where $\phi : I \times R^n \times R^n \to R$ satisfies

$$\int_I \phi(t, x, \dot{x})\,dt - \int_I \left\{ \phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta^T(t)\, G\, \beta(t) \right\} dt$$
$$\geq \int_I \left\{ \eta^T \phi_x(t, \bar{x}, \dot{\bar{x}}) + (D\eta)^T \phi_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \eta^T G\ \beta(t) \right\} dt,$$

then $\int_I \phi(t, x, \dot{x})\,dt$ is second-order invex with respect to η where

$G = \phi_{xx} - 2D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}} - D^3\phi_{\dot{x}\ddot{x}}$, and $\beta \in C(I, R^n)$, the space of n-dimensional continuous vector functions.

Definition 2.2 (Second- order Pseudoinvex) If the functional $\int_I \phi(t, x, \dot{x})\,dt$ satisfies

$$\int_I \left\{ \eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\ \beta(t) \right\} dt \geq 0$$
$$\Rightarrow \int_I \phi(t, x, \dot{x})\,dt \geq \int_I \left\{ \phi(t, \bar{x}, \dot{\bar{x}}) - \frac{1}{2}\beta(t)^T\, G\ \beta(t) \right\} dt,$$

Then $\int_I \phi(t,x,\dot{x})dt$ is said to be second-order pseudoinvex with respect to η.

Definition 2.3 (Second- order Quasi-invex) If the functional $\int_I \phi(t,x,\dot{x})dt$ satisfies

$$\int_I \phi(t,x,\dot{x})dt \leq \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\}dt$$
$$\Rightarrow \int_I \left\{\eta^T\phi_x + (D\eta)^T\phi_{\dot{x}} + \eta^T G(t)\beta(t)\right\}dt \leq 0,$$

then $\int_I \phi(t,x,\dot{x})dt$ is said to be second-order quasi-invex with respect to η.

Remark 2.1 If ϕ does not depend explicitly on *t*, then the above definitions reduce to those given in [2] for static cases.
The following giving Schwartz inequality is also required for the validation of our duality results.

Lemma 2.1 (Schwartz inequality) It states that

$$x(t)B(t)z(t) \leq \left(x(t)^T B(t)x(t)\right)^{1/2}\left(z(t)^T B(t)z(t)\right)^{1/2}, \quad t\in I \tag{1}$$

with equality in (1) if and only if $B(t)\big(x(t)-q(t)z(t)\big)=0$ for some $q(t)\in R$.

Proposition 2.1 If (CP) attains an optimal solution at $x=\bar{x}\in X$, then there exist Lagrange multiplier $\tau\in R$ and piecewise smooth $y: I\to R_+^m$, not both zero, and also piecewise smooth $\hat{\omega}: I\to R^n$, satisfying for all $t\in I$.

$$\tau f_x(t,\bar{x},\dot{\bar{x}}) + B(t)\hat{\omega} + y(t)^T g_x(t,\bar{x},\dot{\bar{x}}) - D\left(\tau f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + y(t)^T g_{\dot{x}}\right) = 0,\ t\in I$$

$$y(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0, t\in I$$

$$\bar{x}(t)^T B(t)\,\bar{\omega}(t) = \left(\bar{x}(t)^T B(t)\,\bar{x}(t)\right)^{1/2}, t \in I$$

$$\bar{\omega}(t)^T B(t)\,\omega(t) \le 1, t \in I$$

The Fritz John necessary optimality conditions given in Proposition7.2.1 become the Karush-Kuhn-Tucker type optimality conditions if $\tau = 1$.

For $\tau = 1$, it suffices that the following slater's conditions [8] holds.

$$g\left(t,\bar{x},\dot{\bar{x}}\right) + g_x\left(t,\bar{x},\dot{\bar{x}}\right)v(t) + g_{\dot{x}}\left(t,\bar{x},\dot{\bar{x}}\right)\dot{v}(t) < 0, \text{For some } v \in X \text{ and all } t \in I.$$

Consider the following class of non-differentiable continuous programming problem studied in [8].

$$(CP): \quad \text{Minimize} \quad \int_I \left(f\left(t,x,\dot{x}\right) + \left(x(t)^T B(t) x(t)\right)^{1/2} \right) dt$$

Subject

$$x(a) = \alpha,\; x(b) = \beta \tag{2}$$

$$g\left(t,x,\dot{x}\right) \le 0, t \in I \tag{3}$$

Where

(i) $I = [a,b]$ is a real interval (ii) $f: I \times R^n \times R^n \to R,\; g: I \times R^n \times R^n \to R^m$ are twice continuously differentiable function with respect to its argument $x(t)$ and $\dot{x}(t)$.

(ii) $x: I \to R^n$ is four times differentiable with respect to t and these derivatives are defined by $\dot{x}, \ddot{x}, \dddot{x}$ and $\ddddot{x}$.

(iii) B (t) is a positive semi-definite $n \times n$ matrix with B (.) continuous on I.

The popularity of mathematical programming problems as (CP) seems to stem from the fact that, even though the objective functions and /or constraint functions are differentiable, a simple formulation of the dual may be given. Non-differentiable mathematical programming deals with much more general kinds of functions by using generalized sub-differentials and quasi differentials.

Husain et al. [5] formulated the following Wolf type second-order dual and established duality results for the pair of problems (CP) and (W-CD) under the second-order pseudoinvexity of $\int_I \left\{ f(t,.,.) + u(.)^T B(t)w(t) + y(t)^T g(t,.,.) \right\} dt$ with respect to η.

(WCD):

Maximize $\int_I \left(f(t,u,\dot{u}) + u(t)^T B(t)w(t) + y(t)^T g_u(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T L\ \beta(t) \right) dt$

Subject to

$$u(a) = 0 = u(b),$$

$$f_u(t,u,\dot{u}) + u(t)^T B(t) + y(t)^T g_u(t,u,\dot{u}) - D\left(f_{\dot{u}} + y(t)^T g_{\dot{u}} \right) + L\ \beta(t) = 0, \quad t \in I,$$

$$y(t) \geq 0, t \in I,$$

$$w(t)^T B(t)w(t) \leq 1, t \in I.$$

where

$$L = f_{uu}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u}) \right)_u - 2D[f_{u\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_u(t,u,\dot{u}) \right)_{\dot{u}}]$$
$$+ D^2[f_{\dot{u}\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right)_{\dot{u}}] - D^3[f_{\ddot{u}\dot{u}}(t,u,\dot{u}) + \left(y(t)^T g_{\dot{u}}(t,u,\dot{u}) \right)_{\ddot{u}}]$$

Recently, Husain and Srivastava [7] investigated Mond-Weir type duality by constructing the following dual to (CP) under $\int_I \left(f(t,u(t),\dot{u}(t)) + w^T B(t) w(t) \right) dt$, for all $w(t) \in R^n$ is second-order pseudo-invex and $\int_I y(t)^T g(t,.,.) dt$ is second-order quasi-invex with respect to the same η.

$(M-WCD)$:

Maximize $\int_I \left\{ f(t,u,\dot{u}) + u(t)^T B(t)w(t) - \frac{1}{2}\beta(t)^T F\ \beta(t) \right\} dt$

Subject to

$$u(a)=0=u(b)$$

$$f_u(t,u,\dot{u})+B(t)w(t)+y(t)^T g_u(t,u,\dot{u})-$$
$$-D\Big(f_{\dot{u}}(t,u,\dot{u})+y(t)^T g_{\dot{u}}(t,u,\dot{u})\Big)+(F+K)\,\beta(t)=0$$

$$\int_I\left(y(t)^T g(t,u,\dot{u})-\frac{1}{2}\beta(t)^T K\,\beta(t)\right)dt\ge 0,\ t\in I$$

$$y(t)\ge 0,\ t\in I$$

$$w(t)^T B(t)w(t)\le 1,\ t\in I$$

where

$$F(t,u,\dot{u},\ddot{u},\dddot{u})=f_{uu}-2Df_{u\dot{u}}+D^2 f_{\dot{u}\dot{u}}-D^3 f_{\dot{u}\ddot{u}},\ t\in I$$

and

$$K(t,u,\dot{u},\ddot{u},\dddot{u})=y(t)^T g_{uu}-2D\,y(t)^T g_{u\dot{u}}+D^2 y(t)^T g_{\dot{u}\dot{u}}-D^3 y(t)\,g_{\dot{u}\ddot{u}},t\in I$$

In this paper, we propose a mixed type second-order dual (MixCD) to (CP) and prove various duality theorems under the assumptions of second-order pseudoinvexity and second-order quasi-invexity. We formulate a pair of nondifferentiable continuous programming problem with natural boundary values rather than fixed end points. Finally, it is also pointed out that our duality results derived in this research can be regarded as the generalization of those of the nonlinear programming problem.

3 Mixed Type second-order duality

In this section we formulate the following mixed type second-order dual Mixed (CD) to (CP):

Mixed (CD):

$$\textit{Maximize}\int_I\left[f(t,u,\dot{u})+u(t)^T B(t)\omega(t)+\sum_{i\in I_0} y^i(t)g^i(t,u,\dot{u})-\frac{1}{2}\beta(t)^T H^0\beta(t)\right]dt$$

Subject to

$$u(a)=0,\ u(b)=0 \tag{4}$$

$$f_u(t,u,\dot{u})+B(t)w)t)+y(t)^T g_u(t,u,\dot{u}) \\ -D\Big(f_{\dot{u}}(t,u,\dot{u})+y(t)^T g_{\dot{u}}(t,u,\dot{u})\Big)+H\,\beta(t)=0, t\in I \tag{5}$$

$$\int_I \left(\sum_{i\in I_\alpha} y^i(t)\, g^i(t,u,\dot{u})-\frac{1}{2}\beta(t)^T G^\alpha\, \beta(t)\right) dt \geq 0.\quad \alpha=1,2,3....r \tag{6}$$

$$w(t)^T B(t)\, w(t)\leq 1,\ \ and\ \ y(t)\geq 0, t\in I \tag{7}$$

Where

$(i)\, I_\alpha \subseteq M=\{1,2,....m\}, \alpha=0,1,2,...r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta=\phi$ if $\alpha\neq\beta$.

$$(ii)\quad H^0 = f_{xx}(t,u,\dot{u})-\sum_{i\in I_0}\Big(y^i(t)g^i(t,u,\dot{u})\Big)_{xx}-2D(f_{x\dot{x}}(t,u,\dot{u}) \\ -\sum_{i\in I_0}\Big(y^i(t)g^i(t,u,\dot{u})\Big)_{x\dot{x}})+D^2(f_{\dot{x}\dot{x}}(t,u,\dot{u})-\sum_{i\in I_0}\Big(y^i(t)g^i{}_{\dot{x}}(t,u,\dot{u})\Big)_{\dot{x}}) \\ -D^3(f_{\dot{u}\ddot{u}}(t,u,\dot{u})-\sum_{i\in I_0}\Big(y^i(t)g^i{}_{\dot{u}}(t,u,\dot{u})\Big)_{\ddot{u}})$$

$$(iii)\quad H = f_{uu}(t,u,\dot{u})-\Big(y(t)^T g_u(t,u,\dot{u})\Big)_u-2D(f_{u\dot{u}}(t,u,\dot{u})-\Big(y(t)^T g_u(t,u,\dot{u})\Big)_{\dot{u}}) \\ +D^2(f_{\dot{u}\dot{u}}(t,u,\dot{u})-\Big(y(t)^T g_u(t,u,\dot{u})\Big)_{\dot{u}}) \\ -D^3(f_{\dot{u}\ddot{u}}(t,u,\dot{u})-\Big(y(t)^T g_u(t,u,\dot{u})\Big)_{\ddot{u}})$$

and

$$(iv)\ G^\alpha=\sum_{i\in I_\alpha}\Big(y^i(t)g_u{}^i(t,u,\dot{u})\Big)_u-2D\left(\sum_{i\in I_\alpha}\Big(y^i(t)g_u{}^i(t,u,\dot{u})\Big)_{\dot{u}}\right) \\ +D^2\left(\sum_{i\in I_\alpha}\Big(y^i(t)g_{\dot{u}}{}^i(t,u,\dot{u})\Big)_{\dot{u}}\right)-D^3\left(\sum_{i\in I_\alpha}\Big(y^i(t)g^i(t,u,\dot{u})\Big)_{\dot{u}}\right)_{\ddot{u}},\ \alpha=1,2,...,r.$$

and

$$(v)\quad G=\sum_{i\in M-I_0}\left(y^i(t)g_u{}^i(t,u,\dot u)\right)_u-2D\left(\sum_{i\in M-I_0}\left(y^i(t)g_u{}^i(t,u,\dot u)\right)_{\dot u}\right)$$
$$+\,D^2\left(\sum_{i\in M-I_0}\left(y^i(t)g_{\dot u}{}^i(t,u,\dot u)\right)_{\dot u}\right)-D^3\left(\sum_{i\in M-I_0}\left(y^i(t)g^i(t,u,\dot u)\right)_{\dot u}\right)_{\ddot u},$$

Theorem 3.1 (Weak Duality)

Let x be feasible for (CP) and (u,y,w,β) feasible for Mixed (CD). If for all feasible (x,u,y,w,β),

$$\int_I\left(f(t,.,.)+\sum_{i\in I_0}y^i(t)g^i(t,.,.)+(.)^T B(t)\,w(t)\right)dt$$

is second-order pseudoinvex and

$$\sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,.,.)\,dt,\alpha=1,2,.....r$$

is second-order quasi-invex with respect to the same η,

then

$$\text{infimum(CP)}\geq\text{supremum Mixed(CD)}$$

Proof: By the feasibility of x and (u,y,w,β) for (CP) and Mixed (CD) respectively, we have

$$\int_I(\sum_{i\in I_\alpha}y^i(t)g^i(t,x,\dot x))dt\leq\int_I(\sum_{i\in I_\alpha}y^i(t)g^i(t,u,\dot u)-\frac{1}{2}\beta(t)^T G^\alpha\beta(t))\,dt,$$

$\alpha=1,2,...,r.$

By second-order quasi-invexity of

$$\sum_{i\in I_\alpha}\int_I y^i(t)g^i(t,x,\dot x)\,dt,\ \alpha=1,2,...,r,$$

this inequality yields

$$\int_I[(\eta^T\sum_{i\in I_\alpha}y^i(t)\,g^i{}_u(t,u,\dot u))+(D\eta)^T(\sum_{i\in I_\alpha}y^i(t)\,g^i{}_{\dot u}(t,u,\dot u))+\eta^T G^\alpha\beta(t)]dt\geq 0,$$

$\alpha = 1, 2, \ldots, r.$

This implies

$$0 \geq \int_I [(\eta^T \sum_{i \in M - I_0} y^i(t)\, g^i{}_u(t,u,\dot{u})) + (D\eta)^T (\sum_{i \in M - I_0} y^i(t)\, g^i{}_{\dot{u}}(t,u,\dot{u})) + \eta^T G\beta(t)]dt$$

$$= \int_I \eta^T \left[\left(\sum_{i \in M - I_0} y^i(t)\, g^i{}_u(t,u,\dot{u}) \right) - D\left(\sum_{i \in M - I_0} y^i(t)\, g^i{}_{\dot{u}}(t,u,\dot{u}) \right) + G\beta(t) \right] dt$$

$$+ \eta^T \left(\sum_{i \in I_0} y^i(t) g^i_u(t,u,\dot{u}) \right) \Bigg|_{t=a}^{t=b}$$

(by integrating by parts)

Using $\eta = 0$, at $t = a$ and $t = b$, we obtain,

$$\int_I \eta^T [(\sum_{i \in M - I_0} y^i(t)\, g^i{}_u(t,u,\dot{u})) - D(\sum_{i \in M - I_0} y^i(t)\, g^i{}_{\dot{u}}(t,u,\dot{u})) + G\beta(t)]dt \leq 0$$

Using (5), we have

$$0 \leq \int_I \eta^T [f_u(t,u,\dot{u}) + B(t)w(t) + \sum_{i \in I_0} y^i(t) g^i_u(t,u,\dot{u}) - D(f_{\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g^i_u(t,u,\dot{u})) + H^0\beta(t)]dt$$

$$= \int_I [\eta^T (f_u(t,u,\dot{u}) + B(t)w(t) + \sum_{i \in I_0} y^i(t) g^i_u(t,u,\dot{u})) + (D\eta)^T (f_{\dot{u}} + \sum_{i \in I_0} y^i(t) g^i_u(t,u,\dot{u})) + \eta^T H^0\beta(t)]dt$$

$$- \eta^T (f_{\dot{u}} + \sum_{i \in I_0} y^i(t) g^i_{\dot{u}}(t,u,\dot{u})) \Bigg|_{t=a}^{t=b}$$

(by integrating by parts)

From this, as earlier $\eta = 0$ at $t = a$ and $t = b$, we get,

$$\int_I [\eta^T (f_u(t,u,\dot{u}) + B(t)w(t) + \sum_{i\in I_0} y^i(t) g^i(t,u,\dot{u}))$$
$$+ (D\eta)^T (f_{\dot{u}}(t,u,\dot{u}) + \sum_{i\in I_0} y^i(t) g^i_{\dot{u}}(t,u,\dot{u})) + \eta^T H^0 \beta(t)]dt \geq 0$$

which by second-order pseudo-invexity of

$$\int_I (f(t,x,\dot{x}) + x^T B(t)w(t) + \sum_{i\in I_0} y^i(t) g^i(t,x,\dot{x}))\,dt$$

implies

$$\int_I (f(t,x,\dot{x}) + x(t)^T B(t)w(t) + \sum_{i\in I_0} y^i(t) g^i(t,x,\dot{x}))\,dt$$
$$\geq \int_I [(f(t,u,\dot{u}) + u(t)^T B(t)w(t) + \sum_{i\in I_0} y^i(t) g^i(t,u,\dot{u})) - \frac{1}{2}\beta(t)^T H^0 \beta(t)]dt$$

Thus from $y(t) \geq 0$ and $g(t,x,\dot{x}) \leq 0,\ t \in I$. The above gives,

$$\int_I \left(f(t,x,\dot{x}) + x(t)^T B(t)w(t) \right) dt$$
$$\geq \int_I [(f(t,u,\dot{u}) + u(t)^T B(t)w(t) + \sum_{i\in I_0} y^i(t) g^i(t,u,\dot{u})) - \frac{1}{2}\beta(t)^T H^0 \beta(t)]dt$$

Since $w(t)^T B(t)\, w(t) \leq 1,\ t \in I$, the Schwartz inequality (1), this inequality implies

$$\int_I (f(t,x,\dot{x}) + \left(x(t)^T B(t) x(t) \right)^{1/2})dt$$
$$\geq \int_I [(f(t,u,\dot{u}) + u(t)^T B(t)w(t) + \sum_{i\in I_0} y^i(t) g^i(t,u,\dot{u})) - \frac{1}{2}\beta(t)^T H^0 \beta(t)]dt$$

yielding

infimum(CP) $\geq$ supremum Mixed (CD)

Theorem 3.2 (Strong Duality)

If $\bar{x}$ is a optimal solution of (CP) and normal [8],then there exist piecewise smooth $y : I \to R^m$ and $w : I \to R^n$ such that $\left(\bar{x}, y, w, \beta(t) = 0\right)$ is feasible for Mixed (CD), and the corresponding values of (CP) and Mix (CD) are equal.

If, for all feasible $\left(\bar{x}, u, y, w, \beta\right)$,

$$\int_I \left\{ f(t,x,\dot{x}) + \sum_{i \in I_0} y^i(t) g^i(t,x,\dot{x}) + x(t)^T B(t) w(t) \right\} dt$$

is second-order pseudo-invex and $\int_I (\sum_{i \in I_\alpha} y^i(t) g^i(t,x,\dot{x}))\, dt$, $\alpha = 1, 2, ..., r$ is second-order quasi-invex, then $\left(\bar{x}, y, w, \beta(t)\right)$ is an optimal solution of Mix (CD).

Proof: Since $\bar{x}$ is an optimal solution of (CP) and normal [8], then by Proposition 2.1, there exist piecewise smooth $y : I \to R^m$ and $w : I \to R^n$ such that

$$f_x\left(t, \bar{x}, \dot{\bar{x}}\right) + B(t) w(t) + y(t)^T g_x\left(t,, \bar{x}, \dot{\bar{x}}\right)$$
$$- D\left(f_{\dot{x}}\left(t, \bar{x}, \dot{\bar{x}}\right) + y(t)^T g_{\dot{x}}\left(t,, \bar{x}, \dot{\bar{x}}\right)\right) = 0, \quad t \in I$$

$$y(t)^T g(t, x, \dot{x}) = 0, \quad t \in I$$

$$\left(\bar{x}(t)^T B(t) \bar{x}(t)\right)^{1/2} = \bar{x}(t) B(t) w(t), t \in I$$

$$w(t)^T B(t) w(t) \le 1, \; t \in I$$

$$y(t) \ge 0, \quad t \in I$$

This implies that $\left(\bar{x}, y, w, \beta(t) = 0\right)$ is feasible for Mix (CD) and the corresponding value of (CP) and Mix (CD) are equal. If

$$\int_I \left\{ f(t,x,\dot{x}) + \sum_{i \in I_o} y^i g^i(t,x,\dot{x}) + + x(t)^T B(t) w(t) \right\} dt$$

is second-order pseudo-invex, and

$$\sum_{i\in I_\alpha}\int_I y(t)^T g(t,x,\dot{x})\,dt,\ \alpha=1,2,....,r$$

is second-order quasi-invex with respect to the same η, then from weak duality Theorem3.1 $(\bar{x},y,w,\beta(t))$ is an optimal solution of Mix (CD).

Theorem 3.3 (Converse Duality) Let (x,y,w,β) be an optimal solution at which

(A_1) for all $\alpha=1,2,3.....r$, either

(a) $\int_I \beta(t)^T\left(G_\alpha+\sum_{i\in I_\alpha}(y^i g^i{}_x)_x\right)\beta(t)\,dt>0$, and $\int_I \beta(t)^T\left(\sum_{i\in I_\alpha}(y^i g^i)_x\right)dt\ge 0,$ *or*

(b) $\int_I \beta(t)^T\left(G^\alpha+\sum_{i\in I_\alpha}\left(y^i g^i_x\right)_x\right)\beta(t)\ dt<0,$ and $\int_I \beta(t)^T\left(\sum_{i\in I_\alpha}(y^i g^i{}_x)_x\right)dt\le 0,$

(A_2) the vectors $H_j^0, G_j^\alpha, \alpha=1,2,...,r$ *and* $j=1,2,...n$ are linearly independent, where H_j^0 is the jth row of the matrix H^0 and G_j^α is the jth row of the matrix G^α and

(A_3) the vectors

$$\sum_{i\in I_\alpha}(y^i(t)g_x^i)-D\sum_{i\in I_\alpha}(y^i(t)g_{\dot{x}}^i),\ \alpha=1,2,3,...,r$$

are linearly independent. If, for all (x,u,y,w,β),

$$\int_I\left\{f(t,...)+(.)^T B(t)w(t)+\sum_{i\in I_\alpha}y^i(t)g^i(t,.,.)\right\}dt$$

is second-order pseudoinvex and

$$\int_I\sum_{i\in I_\alpha}y^i(t)g^i(t,.,.)dt,\ \alpha=1,2,...,r,$$

is second-order quasi-invex, then x is an optimal solution of (CP).

Proof: Since (x,y,u,β) is an optimal solution of Mix (CD), (2), by Proposition 2.1, there $\tau_0\in R,\ \ \tau_\alpha\in R,\ \ \alpha=1,2,...,r,$ and piecewise smooth $\theta:I\to R^m, r:I\to R^m$ and $p:I\to R^n$ such that

$$
\begin{aligned}
0 = \tau_0[(f_x + \sum_{i\in I_0}\left(y^i(t)g_x^i\right) + B(t)w(t)) - D(f_{\dot{x}} + \sum_{i\in I_0}\left(y^i(t)g^i\right)_{\dot{x}}) - \frac{1}{2}\left(\beta(t)^T H^0 \beta(t)\right)_x \\
+\frac{1}{2}D\left(\beta(t)^T H^0\beta(t)\right)_{\dot{x}} - \frac{1}{2}D^2\left(\beta(t)^T H^0\beta(t)\right)_{\ddot{x}} + \frac{1}{2}D^3\left(\beta(t)^T H^0\beta(t)\right)_{\dddot{x}} \\
-\frac{1}{2}D^4\left(\beta(t)^T H^0\beta(t)\right)_{\ddddot{x}}] \\
+\theta(t)^T[(f_{xx} + y(t)^T g_{xx}) - D\left(f_{x\dot{x}} + y(t)^T g_{x\dot{x}}\right) - D\left(f_{x\dot{x}} + (y(t)^T g_{\dot{x}})_x\right) \\
- D\left(D(f_{\dot{x}\dot{x}} + y(t)^T g_{\dot{x}\dot{x}}\right) + D^2\left(D(f_{\dot{x}\ddot{x}} + (y(t)^T g_{\dot{x}})_{\ddot{x}}\right) + (H\beta(t))_x \\
- D(H\beta(t))_{\dot{x}} + D^2(H\beta)_{\ddot{x}} - D^3(H\beta)_{\dddot{x}} + D^4(H\beta)_{\ddddot{x}}] \\
-\sum_{\alpha=1}^{r}\tau_\alpha\{\sum_{i\in I_\alpha}\left(y^i(t)g^i)_x - D(y^i(t)g^i)_{\dot{x}}\right) - \frac{1}{2}(\beta(t)G^\alpha\beta(t)) + \frac{1}{2}D(\beta(t)G^\alpha\beta(t))_{\dot{x}} \\
-\frac{1}{2}D^2(\beta(t)G^\alpha\beta(t))_{\ddot{x}} + \frac{1}{3}D^3(\beta(t)G^\alpha\beta(t)) - \frac{1}{4}D^2(\beta(t)G^\alpha\beta(t))\}, \quad t\in I
\end{aligned} \tag{8}
$$

$$\tau_0(g^i - \frac{1}{2}\beta(t)^T g_{xx}^i\beta(t)) - \theta(t)^T(g_x^i + g_{xx}^i\beta(t)) + r^i(t) = 0, \qquad i\in I_0 \tag{9}$$

$$\tau_0(g^i - \frac{1}{2}\beta(t)^T g_{xx}^i\beta(t)) - \theta(t)(g_x^i + g_{xx}^i\beta(t)) + r^i(t) = 0,\ i\in I_\alpha,\ \alpha = 1,2,\dots,r \tag{10}$$

$$\tau_0 B(t)x(t) - \theta(t)B(t) - 2p(t)\left(B(t)w(t)\right) = 0,\ t\in I \tag{11}$$

$$\left(\tau_0\beta - \theta(t)\right)H^0 + \sum_{\alpha=1}^{r}\left(\tau_\alpha\beta - \theta(t)\right)G^\alpha = 0 \tag{12}$$

$$\tau_\alpha\int_I\{\sum_{i\in I_\alpha}\left(y^i(t)g^i\right) - \frac{1}{2}\beta(t)^T G^\alpha\beta(t)\}dt = 0,\quad \alpha = 1,2,3,\dots,r \tag{13}$$

$$p(t)^T\left(w(t)^T B(t)w(t) - 1\right) = 0,\ t\in I \tag{14}$$

$$r(t)^T y(t) = 0,\ t\in I \tag{15}$$

$$\left(\tau_0,\tau_1,\tau_2,\dots\dots\tau_r, r(t), p(t)\right) \ge 0,\ t\in I \tag{16}$$

$$\left(\tau_0,\tau_1,\tau_2,\dots\dots\tau_r, r(t),\theta(t), p(t)\right) \ne 0,\quad t\in I \tag{17}$$

Because of assumption (A_2), (10) implies

$$\tau_\alpha\beta(t) - \theta(t) = 0, \alpha = 0,1,2,\dots r \tag{18}$$

Multiplying (10) by $y^i(t), t\in I_\alpha,\ \alpha = 1,2,\dots,r,$ and summing over i, we have,

$$\tau_\alpha\{y^i(t)g^i - \frac{1}{2}\beta(t)(y^i g^i)_{xx}\beta(t)\} + \theta(t)\{(y^i g^i)_x + (y^i g^i)_{xx}\beta(t)\} + y^i(t)r^i(t) = 0$$

$$\tau_\alpha\{\sum_{i\in I_\alpha} y^i(t)g^i - \frac{1}{2}\beta(t)^T \sum_{i\in I_\alpha}(y^i g^i)_{xx}\beta(t)\}$$
$$+\theta(t)\{\sum_{i\in I_\alpha}(y^i g^i)_x + \sum_{i\in I_\alpha}(y^i g^i)_{xx}\beta(t)\} = 0,\ i\in I_{\alpha,}\ \ \alpha = 1,2,...,r \qquad (19)$$

Using (18) in (19), we have

$$\tau_\alpha\{\sum_{i\in I_\alpha} y^i(t)g^i - \frac{1}{2}\beta(t)^T \sum_{i\in I_\alpha}(y^i g^i)_{xx}\beta(t)\}$$
$$-\tau_\alpha\beta(t)\{\sum_{i\in I_\alpha}(y^i g^i)_x + \sum_{i\in I_\alpha}(y^i g^i)_{xx}\beta(t)\} = 0,\ i\in I_\alpha,\ \ \alpha = 1,2,...,r. \qquad (20)$$

$$-\tau_\alpha\{\int_I \beta(t)^T \sum_{i\in I_\alpha}(y^i g^i)_x dt + \int_I \beta(t)^T \sum_{i\in I_\alpha}(y^i g^i)_{xx}\beta(t)dt\}$$
$$+\tau_\alpha\int_I [\sum_{i\in I_\alpha}(y^i g^i) - \frac{1}{2}\beta(t)^T(\sum_{i\in I_\alpha}(y^i g^i))_{xx}]\beta(t)\,dt = 0,\ \ \alpha = 1,2,...,r. \qquad (21)$$

gives

$$\tau_\alpha[\int_I \beta(t)(\sum_{i\in I_\alpha}(y^i g^i)_x)dt + \frac{1}{2}\int_I \beta(t)^T(\sum_{i\in I_\alpha}(y^i g^i)_x)\beta(t)\,dt]$$
$$+\frac{\tau_\alpha}{2}\int_I \beta(t)G^\alpha\beta(t)dt = 0 \qquad (22)$$

$$\tau_\alpha\int_I \beta(t)^T \sum_{i\in I_\alpha}(y^i g^i)_x dt + \frac{\tau_\alpha}{2}\int_I \beta(t)(\sum_{i\in I_\alpha}(y^i g^i)_{xx} + G^\alpha)\beta(t)dt = 0 \qquad (23)$$

If for all $\alpha = 0,1,2....r, \tau_\alpha = 0$, then (18) implies $\theta(t) = 0, t\in I$

From (10), we have $r(t) = 0, t\in I$ and $p(t) = 0$ from (11) and (14).

Thus $(\tau_0, \tau_1, \tau_2, ...\tau_r, r(t), \theta(t), p(t)) = 0, t\in I$

This gives a contradiction. Hence there exists an $\bar{\alpha}\in\{0,1,2,.....r\}$ such that $\tau_{\bar{\alpha}} > 0$. If $\beta(t)\neq 0, t\in I$, thus (18) gives $(\tau_\alpha - \bar{\tau}_\alpha)\beta(t) = 0,\ \alpha = 1,2,...,r.$

This implies that $\tau_\alpha = \bar{\tau}_\alpha > 0$, from (20), we have

$$2\int_I \beta(t)^T (\sum_{i\in I_\alpha} (y^i g^i)_x) dt + \int_I \beta(t)^T (G^\alpha + \sum_{i\in I_\alpha} (y^i g^i)_{xx}) \beta(t) dt = 0$$

This contradicts (A_1). Hence $\beta(t) = 0, t \in I$.

Using (4) and $\beta(t) = 0, t \in I, (8)$ *gives*

$$\sum_{\alpha=1}^{r} (\tau_\alpha - \tau_0)\{\sum_{i\in I_\alpha} (y^i g_x^i) - D\sum_{i\in I_\alpha} (y^i g_{\dot{x}}^i)\} = 0, \tag{24}$$

Which by the linear independence of

$$\{\sum_{i\in I_\alpha} (y^i g_x^i) - D\sum_{i\in I_\alpha} (y^i g_{\dot{x}}^i)\}, \quad \alpha = 1, 2, 3, \ldots, r$$

yields

$\tau_\alpha = \tau_{\bar{\alpha}}$, for all $\alpha \in \{0, 1, 2, ...r\}$

Now (9) and (10) gives

$\tau_0 g^i + r_i(t) = 0, t \in I, i \in I_0$ gives $g^i(.) \le 0, i \in I_0$ (25)

$\tau_\alpha g^i + r^i(t) = 0, t \in I, i \in I_\alpha$ implies $g^i(.) \le 0, i \in I_\alpha, \alpha = 1, 2, 3, \ldots r.$ (26)

Gives g (.) ≤ 0 implies x is a feasible solution of (CP).

Multiplying (25) by $y^i(t), i \in I_0$ and to $y^i(t), i \in I_\alpha, \alpha = 1, 2, .., r$

From (11), we have

$$B(t)x(t) = \frac{2p(t)}{\tau_0} B(t)\omega(t)$$

Hence

$$x(t)^T B(t) w(t) = \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \left(w(t)^T B(t) w(t)\right)^{\frac{1}{2}} \tag{27}$$

If $p(t) > 0,\ t \in I$, (14) implies $w(t)^T B(t) w(t) = 1$ and (25) implies

$$x(t)^T B(t) w(t) = \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}},\ t \in I$$

If $p(t) = 0,\ t \in I$, then (27) gives

$$B(t)x(t)=0, t\in I,$$

Thus in either Case, we have

$$x(t)^T B(t)w(t)=\left(x(t)^T B(t)x(t)\right)^{\frac{1}{2}}, t\in I$$

This gives $\int_I \left\{ f(t,x,\dot{x})+\left(x(t)^T B(t)x(t)\right)^{\frac{1}{2}} \right\} dt$

$$=\int_I \left\{ f(t,x,\dot{x})+x(t)^T B(t)w(t)+\sum_{i\in I_0} y^i g^i(t,x,\dot{x})-\frac{1}{2}\beta(t)H^0\beta(t)\right\} dt$$

implies x is optimal to (CP).

4 Special Cases

If $I_0=M$, then Mix (CD) becomes (WD) and from Theorem 3.1-3.3, it follows that it is a second-order dual to (CP), if

$\int_I \left\{ f(t,x,\dot{x})+y^T(t)g(t,x,\dot{x})+x(t)^T B(t)w(t)\right\} dt$ is a second-order pseudoinvex.

If I_0 is empty set and $I_\alpha=M$ for some $\alpha\in\{1,2,...,r\}$, then Mix (CD) reduces to (M-WD) which is a second-order dual to (CP).

If

$$\int_I \left\{ f(t,x,\dot{x})+x(t)^T B(t)w(t)\right\} dt$$

is second-order pseudoinvex and $\int_I y(t)^T g(t,x,\dot{x})dt$, is second-order quasi-invex.

If $B(t)=0, t\in I$, the dual problem Mixed (CD) will reduce to the Mixed type dual treated by Husain and Bilal [9].

5 Natural Boundary Values

In this section, we formulate a pair of nondifferentiable mixed type dual variational problems with natural boundary values rather than fixed end points.

(CP$_0$):

Minimize

$$\int_I \left\{ f\left(t,x,\dot{x}\right) + \left(x(t)^T B(t) x(t)\right)^{\frac{1}{2}} \right\} dt$$

Subject to

$$g\left(t,x,\dot{x}\right) \leq 0, \qquad t \in I$$

(MixCD$_0$):

Maximize

$$\int_I \left(f\left(t,u(t),\dot{u}(t)\right) + u(t)^T B(t) w(t) + \sum_{i\in I_0} y^i(t) g^i\left(t,u,\dot{u}\right) - \frac{1}{2}\beta(t)^T H^0 \beta(t)\right)dt$$

Subject to

$$\left(f_u\left(t,u,\dot{u}\right) + B(t) w(t) + y(t)^T g_u\left(t,u,\dot{u}\right)\right)$$

$$-D\left(f_{\dot{u}}\left(t,u,\dot{u}\right) + y(t)^T g_{\dot{u}}\left(t,u,\dot{u}\right)\right) + H\ \beta(t) = 0, \quad t \in I$$

$$\int_I \left(\sum_{i\in I_\alpha} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T G^\alpha \beta(t)\right)dt \geq 0, \quad \alpha = 1,2,3,\ldots,r$$

$$w(t) B(t) w(t) \leq 1, \ y(t) \geq 0, \quad t \in I$$

$$\sum_{i\in I_\alpha} y^i(t) g_u{}^i\left(t,u,\dot{u}\right) = 0,$$

and

$$f_{\dot{u}}\left(t,u,\dot{u}\right) + \sum_{i\in I_0} y^i(t) g_{\dot{u}}^i\left(t,u,\dot{u}\right) = 0, \ \text{ at } \ t = a \text{ and } t = b.$$

6 Nondifferentiable Nonlinear Programming Problems

If all the functions of the problem (CP_0) and Mix (CD_0) are independent of t as $b-a=1.$ Then these problem will reduces to the following dual problem to (CP_0) formulated by Zhang and Mond [10] :

(CP1): Minimize $f(x)+\left(x^T B x\right)^{1/2}$

Subject to $g(x)\le 0,$

Mix (CD_1): Maximize $f(u)+\sum_{i\in I_0} y^i g^i(u)+u^T B w-\frac{1}{2}\beta^T \widehat{H}^0\beta$

Subject to

$$f_u(u)+Bw+y^T g_u(u)+y^T g_u(u)+\widehat{H}\beta=0$$

$$\sum_{i\in I_0} y^i g_i(u)-\frac{1}{2}\beta^T\hat{G}\beta\ge 0$$

$$w^T B w\le 1,\ y\ge 0$$

where

$$\hat{H}^0=f_{uu}(u)-\left(\sum_{i\in I_0} y^i g^i\right)_{u\dot{u}} \quad \text{and} \quad \hat{G}=\left(\sum_{i\in I_\alpha} y^i g^i\right)_{uu},\ \alpha=1,2,...,r.$$

In Theorem 3.3, the symbols H^0 $,H_j^0,G^\alpha$ *and* G_j^α will respectively become as

$$H^0= f_{xx}(x)-\left(\sum_{i\in I_0} y^i g^i(x)\right)_{xx} =\nabla^2\left(f(x)-\sum_{i\in I_0} y^i g^i(x)\right)$$

$$H_j^0=\left[\nabla^2\left(f(x)-\sum_{i\in I_0} y^i g^i(x)\right)\right]_j,$$

$$G^\alpha=\left(\sum_{i\in I_\alpha} y^i g^i(x)\right)_{xx}=\nabla^2\left(\sum_{i\in I_\alpha} y^i g^i(x)\right) \quad \text{and} \quad G_j^\alpha=\left[\nabla^2\left(\sum_{i\in I_\alpha} y^i g^i(x)\right)\right]_j$$

For the problems (CP_0) and Mix (CD_0) will be stated as the following theorem (Theorem 6.1) established by Yang and Zhang [11].Theorem 6.1 is Theorem 2 of [11] and the rectified version of [10].

Theorem 6.1 Let (x, y, w, β) be an optimal solution of Mix (CD_0) at which

(A_1) for all $\alpha = 1,2,..,r.$ either

(a) the $n \times n$ Hessian matrix $\nabla^2 \sum_{i \in I_\alpha} y^i g^i(x)$ is positive definite, and

$$\beta^T \nabla \sum_{i \in I_\alpha} y^i g^i(x) \geq 0$$

or

(b) the $n \times n$ Hessian matrix $\nabla^2 \sum_{i \in I_\alpha} y^i g^i(x)$ is negative definite and

$$\beta^T \nabla \sum_{i \in I_\alpha} y^i g^i(x) \leq 0,$$

(A_2) The vectors $\left[\nabla^2 \left(f(x) - \sum_{i \in I_0} y^i g^i(x) \right)\right]_j, \left[\nabla^2 \left(\sum_{i \in I_\alpha} y^i g^i(x) \right)\right]_j,$

$\alpha = 1,2,..,r.$ $j = 1,2,...,n$ are linearly independent, where it is the jth row of the matrix H^0 and G^α.

(A_3) the vectors $\{\nabla \sum_{i \in I_\alpha} y^i g^i(x), \ \alpha = 1,2,..,r\}$ are linearly independent.

If, for all feasible (x, u, y, w, p), $f(.) + \sum_{i \in I_\alpha} y^i g^i(.) + (.)^T B w$ is second-order pseudoconvex and $\sum_{i \in I_\alpha} y^i g^i(.), \alpha = 1,2,..,r$ is second-order quasiconcave with respect to the same η, then x is an optimal solution to (CP_0).

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