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Convergence of Proximal Point Algorithms of Mann and Halpern Hybrid Types to a Zero of Monotone Operators in CAT(0) Spaces

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Abstract

In this paper, by the classic Mann-type and Halpern-type algorithms, on the basis of monotone operators with firmly nonexpansive property, we build Mann-Halpern type and Halpern-Mann type proximal point algorithms about a zero of monotone operators in Hadamard space, and prove strong convergence and $\Delta-$convergence to a zero of monotone operators, respectively.

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Keywords: Monotone operators; Mann-Halpern type; Halpern-Mann type; Proximal point algorithms; Hadamard space; $\Delta-$convergence; Strong convergence

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1 Introduction

Let (X, d) be a metric space[11]. A geodesic path joining $x \in X$ to $y \in X$(or, more briefly, a geodesic from x to y) is a map f from a closed interval $[0, l] \subset R$ to X such that $f(0) = x$, $f(l) = y$ and $d(f(t), f(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, f is an isometry and $d(x, y) = l$. The image α of f is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic is denoted $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic space (X, d) is a CAT(0) space if it satisfies the following CN-inequality for $x, z_0, z_1, z_2 \in X$ such that $d(z_0, z_1) = d(z_0, z_2) = \frac{1}{2}d(z_1, z_2)$:

$$d^2(x, z_0) \leqslant \frac{1}{2}d^2(x, z_1) + \frac{1}{2}d^2(x, z_2) - \frac{1}{4}d^2(z_1, z_2).$$

A complete CAT(0) space is called a Hadamard space.

Berg and Nikolaev[3] introduced the concept of quasi-linearization in CAT(0) space X. They denoted a vector by $\overrightarrow{ab}$ for $(a, b) \in X \times X$ and defined the quasi-linearization map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to R$ as follow:

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}[d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)],$$

for $a, b, c, d \in X$. We can verify $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, e \in X$. For a space X, it satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leqslant d(a, b)d(c, d)$$

for all $a, b, c, d \in X$.It is known[3] that a geodesically connected metric space X is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Ahmadi Kakavandi and Amini[1] introduced the concept of dual space of a complete CAT(0) space X based on a work of Berg and Nikolaev[4]. Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy} \rangle := \alpha \langle x^*, \overrightarrow{xy} \rangle + \beta \langle y^*, \overrightarrow{xy} \rangle,$$

for $\alpha, \beta \in R,\ x, y \in X, and\ x^*, y^* \in X^*$, where X^* is the dual space of X.

It is known that the subdifferential of every proper convex and lower semi-continuous function is maximal monotone in Hilbert spaces, and it satisfies the range condition. Ahmadi Kakavandi and Amini[1] also introduced the subdifferential of a proper convex and lower semi-continuous function on a Hadamard space X as a monotone operator from X to X^*.

By the application of the dual theory[1], H.Khatibzadch and S.Ranjbar[2] have showed that the sequences generated by the Mann-type and the Halpern-type proximal point algorithm containing the resolvent of a monotone operator which satisfies range condition are strong convergence and $\Delta-$convergence to a zero of a monotone operator in a complete CAT(0) space, respectively. Hence, we build Mann-Halpern type and Halpern-Mann type proximal point algorithms about zeros of the subdifferential of proper convex and lower semi-continuous function in Hadamard space, and prove strong convergence and $\Delta-$convergence to a zero of a monotone operator, respectively. Therefore, we improve and extend their results.

2 Preliminary

Definition 2.1. [4] *Let $\lambda > 0$ and $A : X \to 2^{X^*}$ be a set-valued operator. The resolvent of A of order λ is the set-valued mapping $J_\lambda : X \to 2^X$ defined by $J_\lambda(x) := \{z \in X : [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}$.*

Definition 2.2. [4] *Let $T : C \subset X \to X$ be a mapping. We say that T is firmly nonexpansive if $d^2(Tx, Ty) \leqslant \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle$ for any $x, y \in C$.*

Let X be a Hadamard space with dual X^* and let $A : X \to 2^{X^*}$ be a multivalued operator with domain $D(A) := \{x \in X : Ax \neq \emptyset\}$, range$R(A) := \bigcup_{x \in X} Ax,\ A^{-1}(x^*) := \{x \in X : x^* \in Ax\}$ and graph $gra(A) := \{(x, x^*) \in X \times X^* : x \in D(A), x^* \in Ax\}$.

Definition 2.3. [4] *Let X be a Hadamard space with dual X^*. The multivalued operator $A : X \to 2^{X^*}$ is:*

(1) monotone if and only if, for all $x, y \in D(A)$, $x^* \in Ax$ and $y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geqslant 0;$$

(2) strictly monotone if and only if for all $x, y \in D(A)$, $x^* \in Ax$ and $y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx} \rangle > 0;$$

(3) $\alpha-$strongly monotone for $\alpha > 0$ if and only if, for all $x, y \in D(A)$, $x^* \in Ax$ and $y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geqslant \alpha d^2(x, y).$$

Definition 2.4. [4] *Let X be a CAT(0) space, $x, y \in X$, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(x, z) = td(x, y)$ and $d(y, z) = (1-t)d(x, y)$. Set $[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}$. A subset C of X is called convex if $[x, y] \subset C$ for all $x, y \in C$.*

Let X be a Hadamard space with dual X^* and let $f : X \to (-\infty, +\infty]$ be a proper function with efficient domain $D(f) = \{x; f(x) < +\infty\}$, then the subdifferential of f is the multifunction $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \{x^* \in X^* : f(z) - f(x) \geqslant \langle x^*, \overrightarrow{xz} \rangle \, (z \in X)\},$$

when $x \in D(f)$ and $\partial f(x) = \emptyset$, otherwise.

Lemma 2.5. [5] *Let (X, d) be a CAT(0) space. Then, for all $x, y, z \in X$, and all $t \in [0, 1]$:*

(1) $d^2(tx \oplus (1-t)y, z) \leqslant td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$,

(2) $d(tx \oplus (1-t)y, z) \leqslant td(x, z) + (1-t)d(y, z)$. In addition, by using (1) we have

$$d\,[tx \oplus (1-t)y, tx \oplus (1-t)z] \leqslant (1-t)d(y, z).$$

Lemma 2.6. [4] *Let (X, d) be a CAT(0) space and $a, b, c \in X$. Then for each $\lambda \in [0, 1]$,*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leqslant \lambda^2 d^2(x, z) + (1-\lambda)^2 d^2(y, z) + 2\lambda(1-\lambda)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle.$$

Lemma 2.7. [7] *Let C be a closed convex subset of a complete CAT(0) space X, $T : C \to C$ be a nonexpansive mapping with a fixed point and $u \in C$. For each $t \in (0,1)$, set $z_t = tu \oplus (1-t)Tz_t$. Then z_t converges as $t \to 0$ to the unique fixed point of T, which is the nearest point to u.*

Lemma 2.8. [6] *Let C be a closed convex subset of a complete CAT(0) space X, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $\{x_n\}$ be a bounded sequence in C such that the sequence $\{d(x_n, Tx_n)\}$ converges to zero. Then*

$$\limsup_n \langle \overrightarrow{up}, \overrightarrow{x_n p} \rangle \leqslant 0,$$

where $u \in C$ and p is the nearest point of $F(T)$ to u.

Lemma 2.9. [4] *Let X be a CAT(0) space and J_λ is resolvent of the operator A of order λ. We have,*

(1) For any $\lambda > 0$, $R(J_\lambda) \subset D(A)$, $F(J_\lambda) = A^{-1}(0)$;

(2) If A is monotone then J_λ is a single-valued and firmly nonexpansive mapping;

(3) If A is monotone and $\lambda \leqslant \mu$, then $d(x, J_\lambda x) \leqslant 2d(x, J_\mu x)$.

It is well known[4] that if T is a nonexpansive mapping on subset C of CAT(0) space X then $F(T)$ is closed and convex. Thus, if A is a monotone operator on CAT(0) space X then, by parts (1) and (2) of Lemma 2.9, $A^{-1}(0)$ is closed and convex.

Lemma 2.10. [8] *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leqslant (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, n \geqslant 0,$$

where , (α_n), (β_n) and (γ_n) satisfy the conditions:

(1) $(\alpha_n) \subset [0,1]$, $\sum_n \alpha_n = \infty$, or equivalently, $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$;

(2) $\limsup_n \beta_n \leqslant 0$;

(3) $\gamma_n \geqslant 0 (n \geqslant 0)$, $\sum_n \gamma_n < \infty$. Then, $\lim_n s_n = 0$.

Lemma 2.11. [9] *Let (γ_n) be a sequence of real numbers such that there exists a subsequence (γ_{n_j}) of (γ_n) such that $\gamma_{n_j} < \gamma_{n_j+1}$ for all $j \geqslant 1$. Then there exists a nondecreasing sequence (m_k) of positive integers such that the following two inequalities:*

$$\gamma_{m_k} \leqslant \gamma_{m_k+1} and \gamma_k \leqslant \gamma_{m_k+1}$$

hold for all (sufficiently large) numbers k. In fact, m_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

By the Lemma 2.6 of S Saejung and P Yotkaew[10], we can similarly obtain the following lemma.

Lemma 2.12. *Let (s_n) be a sequence of nonnegative real numbers, (α_n) be a sequence in $(0, 1)$ such that $\sum_n \alpha_n = \infty$, (t_n) be a sequence of real numbers, and (γ_n) be a sequence of nonnegative real numbers such that $\sum_n \gamma_n < \infty$. Suppose that*

$$s_{n+1} \leqslant (1 - \alpha_n)s_n + \alpha_n t_n + \gamma_n, n \geqslant 1.$$

If $\limsup_{k\to\infty} t_{n_k} \leqslant 0$ for every subsequence (s_{n_k}) of (s_n) satisfying $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geqslant 0$, then $\lim_n s_n = 0$.

Proof. The proof is split into two cases.

(1) There exists an $n_0 \in N$ such that $s_{n+1} \leqslant s_n$ for all $n \geqslant n_0$. It follows then that $\liminf_{n\to\infty}(s_{n+1} - s_n) = 0$. Hence $\limsup_{n\to\infty} t_n \leqslant 0$. The conclusion follows from Lemma 2.10.

(2) There exists a subsequence (s_{m_j}) of (s_n) such that $s_{m_j} < s_{m_j+1}$ for all $j \in N$. In this case, we can apply Lemma 2.11 to find a nondecreasing sequence $\{n_k\}$ of $\{n\}$ such that $n_k \to \infty$ and the following two inequalities:

$$s_{n_k} \leqslant s_{n_k+1} \text{ and } s_k \leqslant s_{n_k+1}$$

hold for all (sufficiently large) numbers k. Since $n_k \to \infty$, then for arbitrary $\varepsilon > 0$, there is a integer $N > 0$ such that $\gamma_{n_k} < \varepsilon$ for $n_k \geqslant N$. It follows from the first inequality that $\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) = 0$. This implies that $\limsup_{k\to\infty} t_{n_k} \leqslant 0$. Moreover, by the first inequality again, we have

$$s_{n_k+1} \leqslant (1 - \alpha_{n_k})s_{n_k} + \alpha_{n_k} t_{n_k} + \gamma_{n_k} \leqslant (1 - \alpha_{n_k})s_{n_k+1} + \alpha_{n_k} t_{n_k} + \varepsilon,$$

this implies $\alpha_{n_k} s_{n_k+1} \leqslant \alpha_{n_k} t_{n_k} + \varepsilon$ for arbitrary $\varepsilon > 0$. By the arbitrariness of ε, we obtain

$$\alpha_{n_k} s_{n_k+1} \leqslant \alpha_{n_k} t_{n_k}.$$

In particular, since each $\alpha_{n_k} > 0$, we have $s_{n_k+1} \leqslant t_{n_k}$. Finally, it follows from the second inequality that

$$\limsup_{k\to\infty} s_k \leqslant \limsup_{k\to\infty} s_{n_k+1} \leqslant \limsup_{k\to\infty} t_{n_k} = 0.$$

Hence $\lim_{n\to\infty} s_n = 0$. This completes the proof. □

Lemma 2.13. [2] *Suppose (X, d) is a metric space and $C \subset X$. Let $(T_n)_{n=1}^{\infty} : C \to C$ be a sequence of nonexpansive mappings with a common fixed point and (x_n) be a bounded sequence such that $\lim_n d(x_n, T_n(x_n)) = 0$. Then*

$$\limsup_n \langle \overrightarrow{up}, \overrightarrow{T_n(x_n)p} \rangle \leqslant \limsup_n \langle \overrightarrow{up}, \overrightarrow{x_n p} \rangle,$$

where $p \in \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.14. [1] *Let $f : X \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function on a Hadamard space X with dual X^*. Then*

(1) f attains its minimum at $x \in X$ if and only if $0 \in \partial f(x)$;

(2) $\partial f : X \to 2^{X^*}$ is a monotone operator;

(3) for any $y \in X$ and $\alpha > 0$, there exist a unique point $x \to X$ such that $[\alpha \overrightarrow{xy}] \in \partial f(x)$.

By the (3) of Lemma 2.14, we obtain the subdifferential of a proper, lower semi-continuous and convex function satisfies the range condition.

Lemma 2.15. [4] *Let $f : X \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function on a Hadamard space X with dual X^*. Then*

$$J_\lambda^{\partial f} x = \underset{z \in X}{Argmin} \{ f(z) + \frac{1}{2\lambda} d^2(z, x) \}$$

for all $\lambda > 0$ and $x \in X$.

Lemma 2.16. [11] *Let K be a closed convex subset of X, and let $f : K \to X$ be a nonexpansive mapping. Then the conditions (x_n) $\Delta-$converges to x and $d(x_n, f(x_n)) \to 0$, imply $x \in K$ and $f(x) = x$.*

3 Main Results

Theorem 3.1. *Let X be a Hadamard space and X^* be the dual space of X. Let $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function, and ∂f is the subdifferential of f. Suppose (λ_n) is a sequence of positive real numbers such that $\lambda_n \geqslant \lambda > 0$, (α_n) is a sequence in $[0,1]$ satisfied $\sum_n \alpha_n < \infty$, and (β_n) is a sequence in $[0,1]$ satisfied $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_n \beta_n = \infty$. The sequence (x_n) generated by the following Mann-Halpern hybrid type algorithm:*

$$\begin{cases} x_0, u \in X, \\ w_n = \underset{x\in X}{argmin}\{f(x) + \frac{1}{2\lambda_n} d^2(x, x_n)\}, \\ y_n = \alpha_n x_n \oplus (1-\alpha_n) w_n, \\ z_n = \underset{y\in X}{argmin}\{f(y) + \frac{1}{2\lambda_n} d^2(y, y_n)\}, \\ x_{n+1} = \beta_n u \oplus (1-\beta_n) z_n. \end{cases} \tag{3.1}$$

Then the sequence is convergent strongly to the nearest point of $\partial f^{-1}(0)$ to u.

Proof. By the Lemma 2.15, the upper algorithm is equivalent to the following algorithm:

$$\begin{cases} x_0, u \in X, \\ y_n = \alpha_n x_n \oplus (1-\alpha_n) J_{\lambda_n} x_n, \\ x_{n+1} = \beta_n u \oplus (1-\beta_n) J_{\lambda_n} y_n, \end{cases} \tag{3.2}$$

where we use J_{λ_n} instead of $J_{\lambda_n}^{\partial f}$.

Since $\partial f^{-1}(0)$ is convex and closed. Set $p \in P_{\partial f^{-1}(0)} u$, we have

$$\begin{aligned} d(x_{n+1}, p) &\leqslant \beta_n d(u,p) + (1-\beta_n) d(J_{\lambda_n} y_n, p) \\ &\leqslant \beta_n d(u,p) + (1-\beta_n)\alpha_n d(x_n, p) + (1-\beta_n)(1-\alpha_n) d(J_{\lambda_n} x_n, p) \\ &\leqslant \beta_n d(u,p) + (1-\beta_n) d(x_n,p) \leqslant \max\{d(u,p), d(x_n,p)\} \\ &\leqslant \cdots \leqslant \max\{d(u,p), d(x_0,p)\}, \end{aligned}$$

which implies that (x_n) is bounded.
Since $d(J_{\lambda_n} x_n, p) \leqslant d(x_n, p)$, then $(J_{\lambda_n} x_n)$ is also bounded.

By the Lemma 2.5, we have

$$
\begin{aligned}
d^2(x_{n+1},p) &= d^2(\beta_n u \oplus (1-\beta_n)J_{\lambda_n}y_n, p) \\
&\leqslant {\beta_n}^2 d^2(u,p) + (1-\beta_n)^2 d^2(J_{\lambda_n}y_n,p) + 2\beta_n(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}y_n p}\rangle \\
&\leqslant {\beta_n}^2 d^2(u,p) + (1-\beta_n)^2({\alpha_n}^2 d^2(x_n,p) + (1-\alpha_n)^2 d^2(J_{\lambda_n}x_n,p)) \\
&\quad + 2(1-\beta_n)^2\alpha_n(1-\alpha_n)\langle \overrightarrow{x_n p}, \overrightarrow{J_{\lambda_n}x_n p}\rangle + 2\beta_n(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}y_n p}\rangle \\
&\leqslant (1-\beta_n)((1-2\alpha_n(1-\alpha_n))d^2(x_n,p)) + {\beta_n}^2 d^2(u,p) \\
&\quad + 2\beta_n(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}y_n J_{\lambda_n}x_n}\rangle + 2\beta_n(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&\quad + 2(1-\beta_n)^2\alpha_n(1-\alpha_n)\langle \overrightarrow{x_n p}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&\leqslant (1-\beta_n)d^2(x_n,p) + \beta_n(\beta_n d^2(u,p) + 2(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&\quad + 2(1-\beta_n)d(u,p)d(y_n,x_n)) + 2\alpha_n\langle \overrightarrow{x_n p}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&\leqslant (1-\beta_n)d^2(x_n,p) + \beta_n(\beta_n d^2(u,p) + 2(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&\quad + 2(1-\beta_n)d(u,p)[\alpha_n d(x_n,x_n) + (1-\alpha_n)d(J_{\lambda_n}x_n,x_n)]) \\
&\quad + 2\alpha_n\langle \overrightarrow{x_n p}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&\leqslant (1-\beta_n)d^2(x_n,p) + \beta_n(\beta_n d^2(u,p) + 2(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&\quad + 2(1-\beta_n)(1-\alpha_n)d(u,p)d(J_{\lambda_n}x_n,x_n)) + 2\alpha_n d(x_n,p)d(J_{\lambda_n}x_n,p),
\end{aligned}
$$

which implies

$$
\begin{aligned}
d^2(x_{n+1},p) \leqslant{}& (1-\beta_n)d^2(x_n,p) + \beta_n(\beta_n d^2(u,p) + 2(1-\beta_n)\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_n}x_n p}\rangle \\
&+ 2(1-\beta_n)(1-\alpha_n)d(u,p)d(J_{\lambda_n}x_n,x_n)) + 2\alpha_n d(x_n,p)d(J_{\lambda_n}x_n,p).
\end{aligned}
$$

By the Lemma 2.12, it suffices to show that $\limsup_{k\to\infty}(\beta_{m_k}d^2(u,p) + 2(1-\beta_{m_k})(1-\alpha_{m_k})d(u,p)d(J_{\lambda_{m_k}}x_{m_k},x_{m_k}) + 2(1-\beta_{m_k})\langle \overrightarrow{up}, \overrightarrow{J_{\lambda_{m_k}}x_{m_k}p}\rangle) \leqslant 0$ for every subsequence $(d(x_{m_k},p))$ of $(d(x_n,p))$ satisfying $\liminf_{k\to\infty}(d(x_{m_k+1},p) - d(x_{m_k},p)) \geqslant 0$. For this, suppose the subsequence $(d(x_{m_k},p))$ satisfied $\liminf_{k\to\infty}(d(x_{m_k+1},p) - d(x_{m_k},p)) \geqslant 0$. Then

$$
\begin{aligned}
0 &\leqslant \liminf_{k\to\infty}(d(x_{m_k+1},p) - d(x_{m_k},p)) \\
&\leqslant \liminf_{k\to\infty}(\beta_{m_k}d(u,p) + (1-\beta_{m_k})d(J_{\lambda_{m_k}}y_{m_k},p) - d(x_{m_k},p)) \\
&\leqslant \liminf_{k\to\infty}(\beta_{m_k}d(u,p) + (1-\beta_{m_k})d(y_{m_k},p) - d(x_{m_k},p)) \\
&\leqslant \liminf_{k\to\infty}(\beta_{m_k}d(u,p) + (1-\beta_{m_k})(\alpha_{m_k}d(x_{m_k},p) \\
&\quad + (1-\alpha_{m_k})d(J_{\lambda_{m_k}}x_{m_k},p)) - d(x_{m_k},p))
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \liminf_{k\to\infty}(\beta_{m_k}(d(u,p)-d(x_{m_k},p)) \\
&\quad + (1-\beta_{m_k})(1-\alpha_{m_k})(d(J_{\lambda_{m_k}}x_{m_k},p)-d(x_{m_k},p))) \\
&\leqslant \limsup_{k\to\infty}(\beta_{m_k}(d(u,p)-d(x_{m_k},p))) \\
&\quad + \liminf_{k\to\infty}(1-\beta_{m_k})(1-\alpha_{m_k})(d(J_{\lambda_{m_k}}x_{m_k},p)-d(x_{m_k},p)) \\
&= \liminf_{k\to\infty}(1-\beta_{m_k})(1-\alpha_{m_k})(d(J_{\lambda_{m_k}}x_{m_k},p)-d(x_{m_k},p)) \\
&\leqslant \limsup_{k\to\infty}(1-\beta_{m_k})(1-\alpha_{m_k})(d(J_{\lambda_{m_k}}x_{m_k},p)-d(x_{m_k},p)) \\
&\leqslant \limsup_{k\to\infty}(1-\beta_{m_k})(1-\alpha_{m_k})(d(x_{m_k},p)-d(x_{m_k},p)) = 0,
\end{aligned}$$

hence, $\lim_{k\to\infty}(d(J_{\lambda_{m_k}}x_{m_k},p)-d(x_{m_k},p))=0$. Since J_{λ_n} is firmly nonexpansive, we have

$$d^2(J_{\lambda_n}x_n,p)\leqslant\langle\overrightarrow{J_{\lambda_n}x_np},\overrightarrow{x_np}\rangle=\frac{1}{2}(d^2(J_{\lambda_n}x_n,p)+d^2(x_n,p)-d^2(J_{\lambda_n}x_n,x_n)),$$

which implies $d^2(J_{\lambda_n}x_n,x_n)\leqslant d^2(x_n,p)-d^2(J_{\lambda_n}x_n,p)$. Then we can get

$$d^2(J_{\lambda_{m_k}}x_{m_k},x_{m_k})\leqslant d^2(x_{m_k},p)-d^2(J_{\lambda_{m_k}}x_{m_k},p),$$

by the boundedness of (x_{m_k}), which implies $d(J_{\lambda_{m_k}}x_{m_k},x_{m_k})\to 0$. By the (3) of Lemma 2.9, we obtain $d(J_\lambda x_{m_k},x_{m_k})\leqslant 2d(J_{\lambda_{m_k}}x_{m_k},x_{m_k})$, which implies $d(J_\lambda x_{m_k},x_{m_k})\to 0$. Therefore, by the Lemma 2.8, we have $\limsup_{k\to\infty}\langle\overrightarrow{up},\overrightarrow{x_{m_k}p}\rangle\leqslant 0$, and by the Lemma 2.13, we obtain

$$\limsup_{k\to\infty}\langle\overrightarrow{up},\overrightarrow{J_{\lambda_{m_k}}x_{m_k}p}\rangle\leqslant 0.$$

Hence, we get $\limsup_{k\to\infty}(\beta_{m_k}d^2(u,p)+2(1-\beta_{m_k})(1-\alpha_{m_k})d(u,p)d(J_{\lambda_{m_k}}x_{m_k},x_{m_k})+2(1-\beta_{m_k})\langle\overrightarrow{up},\overrightarrow{J_{\lambda_{m_k}}x_{m_k}p}\rangle)\leqslant 0$. By the boundedness of $(J_{\lambda_n}x_n)$ and (x_n), we obtain $\sum_n 2\alpha_n d(x_n,p)d(J_{\lambda_n}x_n,p)<\infty$. Hence, by the Lemma 2.13, we know $\lim_{n\to\infty}d(x_n,p)\to 0$. This completes the proof. □

Theorem 3.2. *Let X be a Hadamard space and X^* be the dual space of X. Let $f:X\to(-\infty,+\infty]$ be a proper convex and lower semi-continuous function, and ∂f is the subdifferential of f. Suppose (λ_n) is a sequence of positive real* *numbers such that* $\lambda_n \geqslant \lambda > 0$, (α_n) *is a sequence in* $[0,1]$ *satisfied* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$, *and* (β_n) *is a sequence in* $[0,1]$ *satisfied* $\limsup_{n\to\infty} \beta_n < 1$. *The sequence* (x_n) *generated by the following Halpern-Mann hybrid type algorithm:*

$$\begin{cases} x_0, u \in X, \\ w_n = \underset{x\in X}{argmin}\{f(x) + \frac{1}{2\lambda_n} d^2(x, x_n)\}, \\ y_n = \alpha_n u \oplus (1-\alpha_n) w_n, \\ z_n = \underset{y\in X}{argmin}\{f(y) + \frac{1}{2\lambda_n} d^2(y, y_n)\}, \\ x_{n+1} = \beta_n y_n \oplus (1-\beta_n) z_n. \end{cases} \tag{3.3}$$

Then the sequence is $\Delta-$*convergent to a point* $p \in \partial f^{-1}(0)$.

Proof. By the Lemma 2.15, the upper algorithm is equivalent to the following algorithm:

$$\begin{cases} x_0, u \in X, \\ y_n = \alpha_n u \oplus (1-\alpha_n) J_{\lambda_n} x_n, \\ x_{n+1} = \beta_n y_n \oplus (1-\beta_n) J_{\lambda_n} y_n, \end{cases} \tag{3.4}$$

where we use J_{λ_n} instead of $J_{\lambda_n}^{\partial f}$.

Let $p \in \partial f^{-1}(0)$, we have

$$\begin{aligned} d(x_{n+1}, p) &\leqslant \beta_n d(y_n, p) + (1-\beta_n) d(J_{\lambda_n} y_n, p) \leqslant d(y_n, p) \\ &\leqslant \alpha_n d(u,p) + (1-\alpha_n) d(J_{\lambda_n} x_n, p) \leqslant \alpha_n d(u,p) + (1-\alpha_n) d(x_n, p), \end{aligned}$$

which implies $d(x_{n+1}, p) \leqslant \max\{d(u,p), d(x_0,p)\}$. Hence, (x_n) is a bounded sequence. Since $d(J_{\lambda_n} x_n, p) \leqslant d(x_n, p)$, then $(J_{\lambda_n} x_n)$ is also bounded. Let $\max\{d(u,p), d(x_0,p)\} = M$. By the assuming, for arbitrary $\varepsilon > 0$, there is a integer $N > 0$ such that we have $\alpha_n < \frac{\varepsilon}{M}$ for $n > N$. Therefore, for $n > N$, we obtain

$$d(x_{n+1}, p) \leqslant d(u,p) \cdot \frac{\varepsilon}{M} + d(x_n, p) \leqslant \varepsilon + d(x_n, p).$$

By the arbitrariness of ε, we get $d(x_{n+1}, p) \leqslant d(x_n, p)$, which implies existence of $\lim_n d(x_n, p)$. Hence, we have

$$\begin{aligned}
0 &= \lim_n [d(x_{n+1}, p) - d(x_n, p)] \\
&\leqslant \liminf_n [\beta_n d(y_n, p) + (1-\beta_n) d(J_{\lambda_n} y_n, p) - d(x_n, p)] \\
&\leqslant \liminf_n [\alpha_n d(u, p) + (1-\alpha_n) d(J_{\lambda_n} x_n, p) - d(x_n, p)] \\
&\leqslant \limsup_n [\alpha_n d(u, p) + (1-\alpha_n) d(J_{\lambda_n} x_n, p) - d(x_n, p)] \\
&\leqslant \limsup_n [\alpha_n d(u, p) - \alpha_n d(x_n, p)] \\
&= \limsup_n \alpha_n [d(u, p) - d(x_n, p)] = 0,
\end{aligned}$$

which means $\lim_n [\alpha_n d(u, p) + (1-\alpha_n) d(J_{\lambda_n} x_n, p) - d(x_n, p)] = 0$. Hence, we obtain

$$\lim_n [d(J_{\lambda_n} x_n, p) - d(x_n, p)] = \lim_n \alpha_n [d(J_{\lambda_n} x_n, p) - d(u, p)] = 0.$$

Since J_{λ_n} is firmly nonexpansive, we have

$$d^2(J_{\lambda_n} x_n, p) \leqslant \langle \overrightarrow{J_{\lambda_n} x_n p}, \overrightarrow{x_n p} \rangle = \frac{1}{2}(d^2(J_{\lambda_n} x_n, p) + d^2(x_n, p) - d^2(J_{\lambda_n} x_n, x_n)),$$

which implies $d^2(J_{\lambda_n} x_n, x_n) \leqslant d^2(x_n, p) - d^2(J_{\lambda_n} x_n, p)$. By the boundedness of (x_n) and $(J_{\lambda_n} x_n)$, we get

$$\lim_n d(J_{\lambda_n} x_n, x_n) = 0.$$

Thus, by the (3) of Lemma 2.9, we obtain

$$d(J_\lambda x_n, x_n) \leqslant 2d(J_{\lambda_n} x_n, x_n),$$

which implies $\lim_n d(J_\lambda x_n, x_n) = 0$.

If subsequence (x_{n_j}) of (x_n) is $\Delta-$convergent to $q \in X$, then we have $d(J_\lambda x_{n_j}, x_{n_j}) \to 0$. Hence, since J_{λ_n} is nonexpansive, by the Lemma 2.16, we have $q \in \partial f^{-1}(0)$. This completes the proof. □

The following theorem shows that the sequence is $\Delta-$convergent for classic Ishikawa type algorithm.

Theorem 3.3. *Let X be a Hadamard space and X^* be the dual space of X. Let $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function, and ∂f is the subdifferential of f. Suppose (λ_n) is a sequence of positive real numbers such that $\lambda_n \geqslant \lambda > 0$, and (α_n), (β_n) are two sequences in $[0,1]$ satisfied $\limsup_{n\to\infty} \alpha_n < 1$ and $\limsup_{n\to\infty} \beta_n < 1$, respectively. The sequence (x_n) generated by the following Ishikawa type algorithm:*

$$\begin{cases} x_0, u \in X, \\ w_n = \underset{x\in X}{argmin}\{f(x) + \frac{1}{2\lambda_n} d^2(x, x_n)\}, \\ y_n = \alpha_n x_n \oplus (1-\alpha_n) w_n, \\ z_n = \underset{y\in X}{argmin}\{f(y) + \frac{1}{2\lambda_n} d^2(y, y_n)\}, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) z_n. \end{cases} \tag{3.5}$$

Then the sequence is $\Delta-$convergent to a point $p \in \partial f^{-1}(0)$.

Proof. It is similar to Theorem 3.2. □

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