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Option valuation pricing model with fuzzy volatility depending on time

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Abstract

We propose a valuation of European call option with fuzzy volatility depending on time. The principle in this valuation where other parameters of option pricing model are supposed to be non fuzzy, consists in replacing volatility by its central value as defined by Bodjanova (see Bojanova 2005 [13]). After having given a sufficient condition guaranteeing the equality of the exact price of European call option with its price when fuzzy volatility is replaced by its central value, a case study is carried out to show the application of the approach suggested.

Mathematics Subject Classification: 03B52, 28E10

Keywords: Option pricing model, fuzzy number, volatility, fuzzy number central value.

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1 Introduction

The theory of option pricing started its long history back in 1900 when the French mathematician Louis Bachelier deduced an option pricing formula in his doctoral thesis (see [2]). He was the first who had the innovative idea of using a stochastic process as a model for the price evolution of stocks. However, option pricing theory really made progress in 1973 (see [3] and [11]) when Fisher Black and Myron Scholes, in cooperation with Robert Merton, derived an option pricing formula that only depends on the actual stock price, the exercise (strike) price, the risk-free interest rate, the volatility of the stock and the expiry date (maturity).

The Black-Scholes model (see for example [16], [7] and [15]) for pricing options at time $t = t^*$ is based on a stochastic differential equation (SDE) of the form

$$dX(t) = X(t)\left[\mu\left(t, X(t)\right)dt + \sigma\left(t, X(t)\right)dB(t)\right], \qquad (1.1)$$

where $X(t) \geq 0$ denotes the asset price that is a function of time t. In particular (see [16]), we consider the time interval $t^* \leq t \leq T$ with an actual time $t^* \geq 0$ and $T > 0$ being the maximum duration of associated European options written on the asset. It is clear that only the actual asset price $X^* \equiv X(t^*)$ and historical data $X(t)$ $(t < t^*)$ are available. The functions $\mu(X, t)$ and $\sigma(X, t)$ are the drift and the local volatility of the stochastic price process $X(t)$ for the asset. The symbol $B(t)$ designates the standard Wierner process (see [8]).

It is known (see [16]) that by virtue of no arbitrage (i.e. $\mu(t, X) = r > 0$ is the constant risk-free interest rate), the actual price $C(X(t),t)$ at time t of an option with maturity T on the asset with actual price $X(t)$ satisfied the Black-Scholes partial differential equation (BSPDE)

$$\frac{\partial C(X,t)}{\partial t}+\frac{1}{2}X^2\sigma^2(X,t)\frac{\partial^2 C(X,t)}{\partial X^2}+rX\frac{\partial C(X,t)}{\partial X}-rC(X,t)=0 \qquad (1.2)$$

The equation (1.2) combined with the terminal condition

$$C(X,t)=h(X), \qquad (1.3)$$

where the payoff function attains the form

$$h(X)=\begin{cases}\max(X-K,0) & \text{for call options,}\\ \max(K-X,0) & \text{for put options,}\end{cases} \qquad (1.4)$$

with strike price K constitutes the option pricing model. Frequently (see [16]), the natural boundary conditions

$$C(0,t)=\begin{cases}0\ (0\le t\le T) & \text{for call options,}\\ K\ (0\le t\le T) & \text{for put options,}\end{cases} \qquad (1.5)$$

are also imposed.

The option pricing model involves five parameters: X, T, K, r and σ. Except for the volatility σ, all others are directly observable parameters (see [8]).

When $\sigma(X,t)\equiv\sigma$ is constant, the famous formula of Black-Scholes (see [3]) gives the value of call price $C(X,t)$ of European call option. However, the actual volatilities can not be expected to be constants where one obvious evidence is the implied volatility smile. The volatilities do vary from time to time and exhibit uncertainty properties (see [14]).

In this paper, we consider the option pricing model (1.2)-(1.3) for a European call option when $\sigma(X,t) \equiv \sigma(t)$ is only function of time t and fuzzy. When $\sigma(t)$ is non fuzzy (see [8]), the value of call price $C(X,t)$ is given by (1.6)

$$C(X,t) = XN(d_1) - Ke^{-r\tau}N(d_2),$$

where

$$d_1 = \frac{\ln\frac{X}{K} + r\tau + \frac{1}{2}\int_0^{\tau} \sigma^2(u)du}{\sqrt{\int_0^{\tau} \sigma^2(u)du}}, \qquad (1.7)$$

$$d_2 = d_1 - \sqrt{\int_0^{\tau} \sigma^2(u)du}\ , \qquad (1.8)$$

N denotes the cumulative distribution function of a standard normal variable and $\tau = T - t$ denotes the time for expiration.

Incompleteness is one facet of uncertainty and randomness is the other, and since there may not be enough data available to develop a probabilistic distribution for $\sigma(t)$, it becomes imperative then to use fuzzy set theory to model the uncertainty of $\sigma(t)$ in (1.6) in order to obtain a more realistic value of call price $C(X,t)$ i.e. our objective is to use fuzzy set theory in order to have a better real approximation of the value of $\sigma(t)$. In other words, one will have

$$\begin{cases} C(X,t) = XN(d_1) - Ke^{-r\tau}N(d_2) \\ \sigma(t) \simeq \tilde{\sigma}(t) \end{cases}, \tag{1.9}$$

where $\tilde{\sigma}(t)$ is real approximation of a fuzzy number $\sigma(t)$.

In the literature, this type of approach was already used for constant volatility (see for example [14]).

The plan of this paper is as follows. In section 2, we summarize some basic results of fuzzy set theory allowing us to approximate a fuzzy number by a real number. In section 3 the valuation of European call option with fuzzy volatility depending on time is carried out. Finally in section 4, we carry out a case study to show the relevance of the method suggested.

2 Basic results of fuzzy set theory

In this section, we introduce certain terminologies, notations and definitions that will be used in the sequel.

Definition 2.1 *Let X be a universal set and A a subset of X. The fuzzy set A is a set of ordered pairs*

$$A = \{(x, \mu_A(x)) | x \in X\} \tag{2.1}$$

where

$$\mu_A : X \to [0,1] \tag{2.2}$$

is a mapping where the range $\mu_A(x)$ of $x \in X$ is called the membership function *or grade of membership (also degree of compatibility or degree of truth) of* x *in* A.

Definition 2.2 *Let* A *be a fuzzy set in* X. *The support of* A, *denoted by* $S(A)$, *is the crisp set of all* $x \in X$ *such that* $\mu_A(x) > 0$.

Definition 2.3 *The α - level set (or α - cut) of a fuzzy set* A *of* X *is a classical set (or crisp interval) denoted by* A_α *and defined as:*

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}. \tag{2.3}$$

Definition 2.4 *Let* A *be a fuzzy set in* X. *The height* $h(A)$ *of* A *is defined as:*

$$h(A) = sup\ \mu_A(x). \tag{2.4}$$

If $h(A) = 1$ then the fuzzy set A is called a *normal fuzzy set*.

Definition 2.5 Let A be a fuzzy set in X. A is *convex* if and only if

$$\mu_A[\lambda x_1 + (1-\lambda)x_2] \geq \min[\mu_A(x_1), \mu_A(x_2)] \tag{2.5}$$

for all x_1, x_2 in X and all $\lambda \in [0,1]$.

Definition 2.6 A *fuzzy number* (or more generally a fuzzy quantity) N is a convex and normal fuzzy set of the real line $\mathbb{R}$.

Remark 2.7 In many situations we often summarize numerical information, as for example, around Rs. 5000, near zero, about ten degrees Celsius, about 15 − 20 percent, possibly not less than 2000 units. These sorts of numerically transmittable data are not precise or crisp in terms of classical mathematical reasoning, but are very meaningful in terms of human communication, perception and reasoning. These imprecise data could be the examples of what are called fuzzy numbers (see [1]).

The membership function of a fuzzy number N has the following properties (see [1]):

(i) $\mu_N(x) = 0,$ outside of some interval $[a, d]$.

(ii) There are real numbers b and c, $a \leq b \leq c \leq d$ such that $\mu_N(x)$ is monotone increasing on the interval $[a, b]$ and monotone decreasing on the interval $[c, d]$.

(iii) $\mu_N(x) = 1,$ for each $x \in [b, c]$.

Thus, in accordance with Bodjanova (see S. Bodjanova 2005 [13]), we assume that the membership function of a fuzzy number N can be expressed for all $x \in \mathbb{R}$ in the form

$$\mu_N(x) = \begin{cases} g(x) \text{ when } x \in [a, b) \\ 1 \text{ when } x \in [b, c] \\ h(x) \text{ when } x \in (c, d] \\ 0 \text{ otherwise,} \end{cases} \tag{2.6}$$

where a, b, c, d are real numbers such that $a < b \leq c < d$, g is a real valued function that is increasing and is right continuous and h is a real valued function that is decreasing and is left continuous. A fuzzy number N with shape functions g and h defined by

$$g(x) = \left(\frac{x-a}{b-a}\right)^n \tag{2.7}$$

$$h(x) = \left(\frac{d-x}{d-c}\right)^n \tag{2.8}$$

respectively, where $n > 0$, will be denoted by $N = \langle a, b, c, d \rangle_n$.

If N is a fuzzy number, then N_α is a closed interval of $\mathbb{R}$ for all $\alpha \in [0,1]$, and, one introduces an alternative notation of N_α as (see [1])

$$N_\alpha = [m_1(\alpha), m_2(\alpha)] \in \mathbb{R} \tag{2.9}$$

where $m_1(\alpha)$ and $m_2(\alpha)$ are respectively the lower and upper bounds of the interval N_α.

Proposition 2.8 [13]
If $N = \langle a, b, c, d \rangle_n$ then for all $\alpha \in [0,1]$,

$$N_\alpha = [g^{-1}(\alpha), h^{-1}(\alpha)] \tag{2.10}$$

Cardinality of a fuzzy number N described by (2.6) is the value of the integral

$$CardN = \int_a^d \mu_N(x)dx = \int_0^1 (m_2(\alpha) - m_1(\alpha))d\alpha. \tag{2.11}$$

Proposition 2.9 [13]

If $N = \langle a, b, c, d \rangle_n$ then

$$CardN = \frac{b-a}{n+1} + (c-b) + \frac{d-c}{n+1}. \qquad (2.12)$$

Definition 2.10 [13]

The median value of a fuzzy number N characterized by (2.6) is the real number M_N from the support of N such that

$$\int_a^{M_N} \mu_N(x)dx = \int_{M_N}^d \mu_N(x)dx \qquad (2.13)$$

For practical purposes (see [13]) expression (2.13) can be rewritten as

$$\int_a^{M_N} \mu_N(x)dx = \frac{CardN}{2} \qquad (2.14)$$

One can classify fuzzy numbers with respect to the distribution of their cardinality as follows (see [13]): a fuzzy number N is called

(i) a fuzzy number with equally heavy tails if $\int_a^b \mu_N(x)dx = \int_c^d \mu_N(x)dx$,

(ii) a fuzzy number with light tails if $\max\left\{\int_a^b \mu_N(x)dx, \int_c^d \mu_N(x)dx\right\} \leq \frac{1}{2}\int_a^d \mu_N(x)dx$,

(iii) a fuzzy number with a heavy left tail if $\int_a^b \mu_N(x)dx > \frac{1}{2}\int_a^d \mu_N(x)dx$,

(iv) a fuzzy number with a heavy right tail if $\int_c^d \mu_N(x)dx > \frac{1}{2}\int_a^d \mu_N(x)dx$.

Let us identify the fuzziness of M_N by its membership grade $\mu_N(M_N)$. We then have the following propositions (see [13]):

Proposition 2.11 If N is a fuzzy number with light tails then

$$M_N = \frac{b+c}{2} + \frac{1}{2}\left(\int_c^d \mu_N(x)dx - \int_a^b \mu_N(x)dx\right) \tag{2.15}$$

and $\mu_N(M_N) = 1$.

Proposition 2.12 Let $N = \langle a, b, c, d \rangle_n$ then

$$M_N = a + \left(\frac{(b-a)^n}{2}(n+1)CardN\right)^{\frac{1}{n+1}} \tag{2.16}$$

if N has a heavy left tail, and

$$M_N = d - \left(\frac{(d-c)^n}{2}(n+1)CardN\right)^{\frac{1}{n+1}} \tag{2.17}$$

if N has a heavy right tail.

Definition 2.13 The center of gravity g_N of the support of a fuzzy number N weighted by the membership grade is given by

$$g_N = \frac{\int_a^d x\mu_N(x)dx}{\int_a^d \mu_N(x)dx} \tag{2.18}$$

Definition 2.14 The center of the core (the central modal value) MO_N of a fuzzy number N is given by

$$MO_N = \frac{b+c}{2} \tag{2.19}$$

Remark 2.15 The value of MO_N does not take into account the shape of the membership function of N. Fuzzy numbers with the same core have the same center of core regardless of their tails. Therefore MO_N represents only the crisp part of N. The value of g_N takes into account the entire membership function of N [13].

Definition 2.16 [13]

Let N be a fuzzy number. Let g_N, MO_N and M_N be the center of gravity, the center of core and the median value of N, respectively. Then the central value of N is given by

$$C_N = \frac{g_N \cdot \mu_N(g_N) + MO_N \cdot \mu_N(MO_N) + M_N \cdot \mu_N(M_N)}{\mu_N(g_N) + \mu_N(MO_N) + \mu_N(M_N)} \tag{2.20}$$

Remark 2.17 The central value can be used as a crisp (i.e. classical) approximation of a fuzzy number (see [13]).

3 Valuation of European call option with fuzzy volatility depending on time

In this section, we consider the pricing option model

$$\frac{\partial C(X,t)}{\partial t} + \frac{1}{2} X^2 \sigma^2(X,t) \frac{\partial^2 C(X,t)}{\partial X^2} + rX \frac{\partial C(X,t)}{\partial X} - rC(X,t) = 0. \tag{3.1}$$

With the payoff

$$C(X,T) = \max(X - K, 0) \tag{3.2}$$

and the fuzzy local volatility function $\sigma(t)$ defined on $\Omega = [0,T]$. All the other parameters are supposed to be non fuzzy.

We always assume $\sigma(t) \in C^{\lambda}, 0 < \lambda \leq 1$, i.e. the fuzzy local volatility is holder continuous in Ω with holder index λ and it satisfies for all $(t, t_0) \in \Omega^2$ and inequality of the form

$$|\sigma(t) - \sigma(t_0)| \leq L|t - t_0|^{\frac{\lambda}{2}}. \quad (3.3)$$

Moreover, we take the fuzzy local volatility function σ strictly positive and bounded with $0 < \sigma_{min} \leq \sigma(t) \leq \sigma_{max} < \infty$.

Let us consider for all $t \in \Omega$,

$$\sigma(t) = \langle 0, \sigma_{min}, \sigma_{max}, \delta \rangle_n \quad (n > 0) \quad (3.4)$$

where $\delta = \sigma_{max} + 10^{-n}$ is a real such that $\sigma_{max} < \delta$.

The main results of this part are the following:

Theorem 3.1 The central value C_σ of σ is given by

$$C_\sigma = \frac{(2n+1)(\sigma_{min} + \sigma_{max}) + \delta}{4(n+1)} \quad (3.5)$$

Lemma 3.2 Let

$C(X(t), t)$, the actual price at time t of a European call option with exact fuzzy volatility depending on time $\sigma(t)$, with maturity T on the asset, with actual price

$X(t)$.

$\tilde{C}(X(t),t)$, the actual price at time t of a European call option with approximate fuzzy volatility (central value) C_σ, with maturity T on the asset, with actual price $X(t)$.

If $n > 0$ is close to 0 then

$$C(X(t),t) = \tilde{C}(X(t),t) \quad (3.6)$$

Proof of theorem 3.1

Taking account of (2.6), (2.7) and (2.8), the membership of σ is given by

$$\mu_\sigma(x) = \begin{cases} \left(\frac{x}{\sigma_{min}}\right)^n \; when \;\; x \in [0, \sigma_{min}) \\ 1 \; when \; x \in [\sigma_{min}, \sigma_{max}] \\ \left(\frac{\delta - x}{\delta - \sigma_{max}}\right)^n when \; x \in (\sigma_{max}, \delta] \\ 0 \text{ otherwise,} \end{cases} \quad (3.7)$$

A simple integral calculus makes it possible to see that σ is a fuzzy number with light tails. Consequently, taking into account proposition 2.11, it is deduced that the median value of σ is:

$$M_\sigma = \frac{n(\sigma_{min} + \sigma_{max}) + \delta}{2(n+1)} \quad (3.8)$$

and that $\mu_\sigma(M_\sigma) = 1$.

A simple calculation allows the taking into account of (2.18) to have the center of gravity

$$g_\sigma = \frac{n(n+1)({\sigma^2}_{max} - {\sigma^2}_{min}) + 2\delta^2 + 2\delta n \sigma_{max}}{2(n+2)[\delta + n(\sigma_{max} - \sigma_{min})]} > \delta \qquad (3.9)$$

and consequently $\mu_\sigma(g_\sigma) = 0$.

Taking into account (2.19), the central modal value of σ is given by

$$MO_\sigma = \frac{\sigma_{min+\sigma_{max}}}{2} \qquad (3.10)$$

and consequently $\mu_\sigma(MO_\sigma) = 1$.

(3.8), (3.9) and (3.10) as well as the values of $\mu_\sigma(M_\sigma)$, $\mu_\sigma(g_\sigma)$, $\mu_\sigma(MO_\sigma)$

carried to (2.20) allow us to have the result.

Proof of lemma 3.2

Let us set $S(T) = \int_0^\tau \sigma^2(u)du$ and $\tilde{S}(T) = \int_0^\tau {C_\sigma}^2(u)du$ where

$\tau = T - t$

with $0 \leq t \leq T$.

Taking into account (1.6), (1.7) and (1.7), it is enough to show that $S(T) = \tilde{S}(T)$.

when n is close to 0, a simple calculation makes it possible to have

$$\tilde{S}(T) = \frac{1}{16}(\sigma_{min} + \sigma_{max} + \delta)^2\tau. \qquad (3.11)$$

(3.11) joints to the fact that $0 < \sigma_{min} \le \sigma(t) \le \sigma_{max} < \delta$ leads to:

$$\frac{1}{4}\sigma^2{}_{max}\tau \le \tilde{S}(T) \le (\sigma_{min} + \sigma_{max} + \delta)^2\tau. \qquad (3.12)$$

In the same way, a simple calculation of framing leads to

$$-(\sigma_{min} + \sigma_{max} + \delta)^2\tau \le \sigma^2{}_{min}\tau \le S(T) \le \sigma^2{}_{max} \qquad (3.13)$$

By adding member with member the framings with $S(T)$ and $\tilde{S}(T)$ one obtains the result.

4 Case study

In this section, we proceed to a case study in order to show the application of the approach presented in the preceding section. For that, we compare the results obtained in calculations of the prices of options for exact function volatility given to those where the function volatility in question is replaced by its central value.

Provided the local volatility function $\sigma(t)$ ($0 \le t \le T$), it is well known (see [8], [16]) that the fair price $C(X,t)$ of European call can be obtained by

$$C(X,t) = X(0)N(d_1) - Ke^{-r\tau}N(d_2), \qquad (4.1)$$

where $\tau = T - t$

$$d_1 = \frac{\ln\frac{X(0)}{K} + r\tau + \frac{1}{2}\int_0^T \sigma^2(u)du}{\sqrt{\int_0^T \sigma^2(u)du}}, \qquad (4.2)$$

$$d_2 = d_1 - \sqrt{\int_0^{\tau} \sigma^2(u)du} \;, \qquad (4.3)$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx \qquad (4.4)$$

is the standard normal distribution function.

For our case of study, we use the same function volatility and the same data as in [16]. We consider as an exact volatility the function

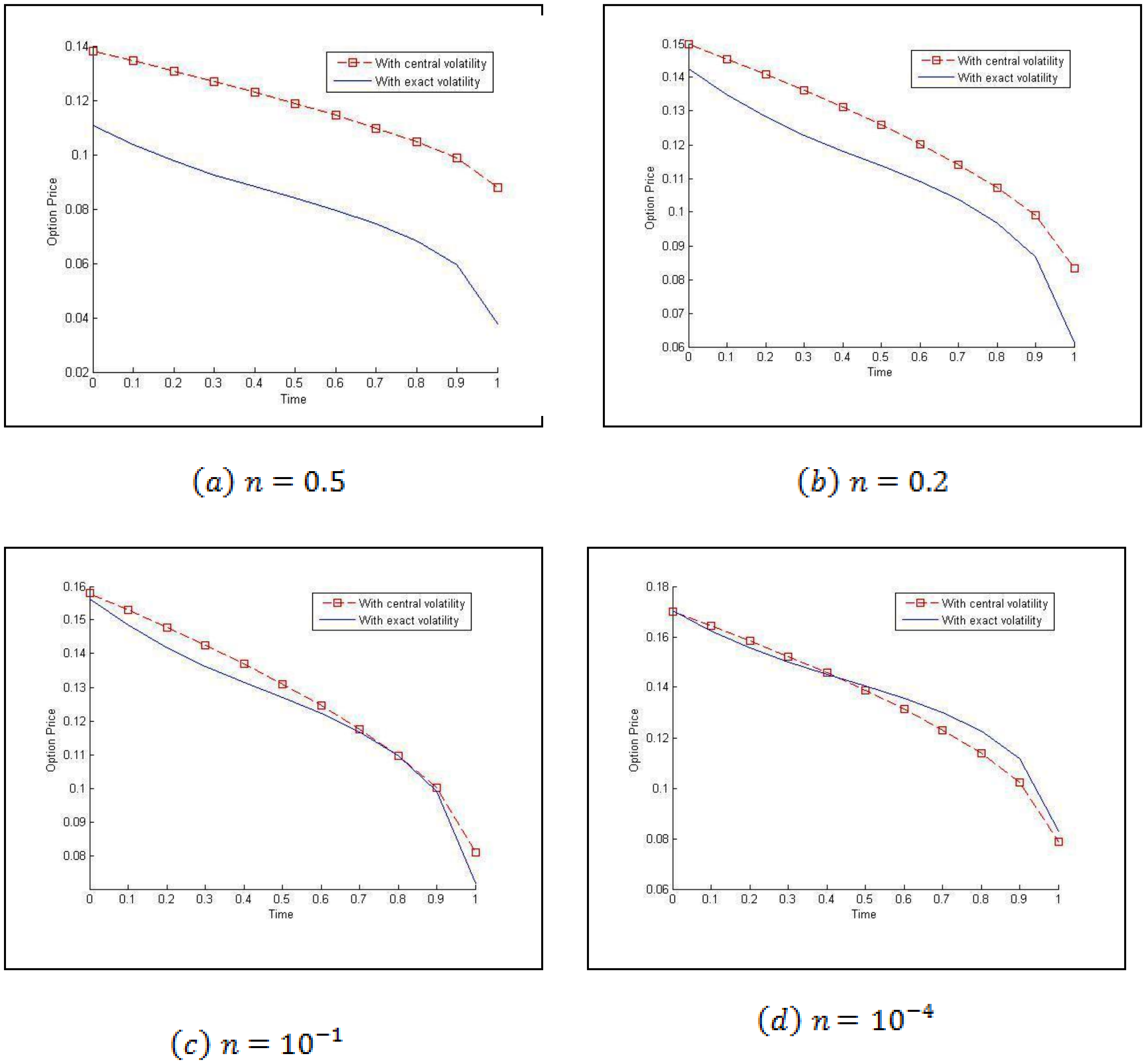
$$\sigma(t) = (t - 0.5)^2 + 0.1 \;\; (0 \le t \le 1) \qquad (4.5)$$

Furthermore, we set for the actual asset price $X(0) = 0.6$, the exercise price $K = 0.5$ for the interest rate $r = 0.05$.

We thus have $\sigma_{min} = \min_{0 \le t \le 1} \sigma(t) = 0.1,\; \sigma_{max} = \max_{0 \le t \le 1} \sigma(t) = 0.35$ and the membership of $\sigma(t)$ $(0 \le t \le 1)$ is given by (see (3.7))

$$\mu_\sigma(x) = \begin{cases} (10x)^n \text{ when } x \in [0, 0.1) \\ 1 \text{ when } x \in [0.1, 0.35] \\ \left(\frac{\delta - x}{\delta - 0.35}\right)^n \text{ when } x \in (0.35, \delta] \\ 0 \text{ otherwise,} \end{cases} \qquad (4.6)$$

The figure 1 presents the curve of the price given by (4.1) of the option for the exact value (4.5) of the function volatility and the curves of the prices given by (4.1) of the option for different values from n when this volatility (4.5) is replaced by its central value (3.5). One notes thus that the two curves converge when n is taken to be very small i.e. close to 0.



$(a)\ n = 0.5$ $(b)\ n = 0.2$

$(c)\ n = 10^{-1}$ $(d)\ n = 10^{-4}$

Figure 1: Comparative curves of the prices of options with central value and exact volatility

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