Journal of Applied Mathematics & Bioinformatics, vol.8, no.2, 2018, 29-46
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2018

# Detecting a layered ellipsoid solving a near-field inverse acoustic scattering problem

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**Abstract**

The scattering problem of time-harmonic acoustic plane waves by a two-layered object consisting of a penetrable triaxial ellipsoid with a soft confocal ellipsoidal core is considered. A low-frequency formulation of the direct scattering problem as well as a Rayleigh approximation is described. Considering near-field data, an inverse acoustic scattering problem is formulated and studied. A finite number of measurements of the leading order term of the scattered field in low-frequency approximation leads to specify the semi-axes of the ellipsoids. The orientation of the ellipsoids is obtained by using the Euler angles. Solving a near-field inverse acoustic scattering problem for the ellipsoids gives results that can also be used for spheroids, spheres, needles and discs, considering them as geometrically degenerate cases of the ellipsoid for appropriate values of the physical and geometrical parameters.

**Mathematics Subject Classification(2010):** 35P25; 78A46; 33E05

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Article Info: *Received* : January 7, 2018. *Revised* : March 2, 2018.
*Published online* : September 15, 2018

**Keywords:** inverse scattering problem; ellipsoidal scatterer; near-field data

## 1 Introduction

The inverse scattering theory is of great importance since it has many applications in medical imaging, detection of buried and underwater objects and geological studies. An ellipsoid can be considered as a good approximation of many objects such as spheres, needles and spheroids. Therefore, the study of inverse scattering problems for ellipsoids is significant. In these problems all we need to specify are the three semi-axes as well as the three Euler angles that fix the position of the principal axes of the ellipsoid.

The case of scattering by a triaxial ellipsoid was studied by Rayleigh in [14]. The inverse scattering problem for an acoustically soft ellipsoid was first studied by Dassios [7] using far-field data. In particular, in this method the size and the orientation of the ellipsoids may be obtained from a finite number of measurements of the leading order term in the low-frequency expansion of the far-field pattern. Later, the cases of the rigid [13] and the penetrable ellipsoid [11] for acoustic scattering problems as well as the ellipsoidal perfect conductor [12] and the dielectric ellipsoid [3] for electromagnetic scattering problems were also studied.

In the present paper, we extend this method for the case of a penetrable triaxial ellipsoid with an acoustically soft confocal ellipsoidal core using near-field data. In near-field inverse scattering problems we use the scattered field and therefore we can avoid the computations of the far-field pattern. Specifically, in this paper, we use the zeroth and the first low-frequency approximations of the low-frequency expansion of the scattered field [6] to specify the orientation and the size of such an ellipsoid.

In section 2 we formulate the direct scattering problem for a penetrable ellipsoid with an acoustically soft confocal core. In section 3 we study the inverse scattering problem by constructing a measurement matrix whose elements are given in terms of measurements taken at four different points for three different directions of propagation and whose eigenvalues and eigenvectors hold information about the orientation and the size of the ellipsoid. In section 4 we study a modified method in the case that the semi-focal distances are known.

In section 5 we study a physically degenerate form of this problem which is the acoustically soft ellipsoid. Finally, in section 6 we study the sphere and the needle as geometrically degenerate forms of the ellipsoid.

## 2 The direct problem

The layered ellipsoid consists of many surfaces of confocal ellipsoids. In this paper we shall describe the proposal method for the case of a two-layered ellipsoid. The same steps are followed for the case of many layers. In particular, we consider the acoustically penetrable ellipsoid with a soft confocal ellipsoid which is the core of the scatterer (Figure 1).

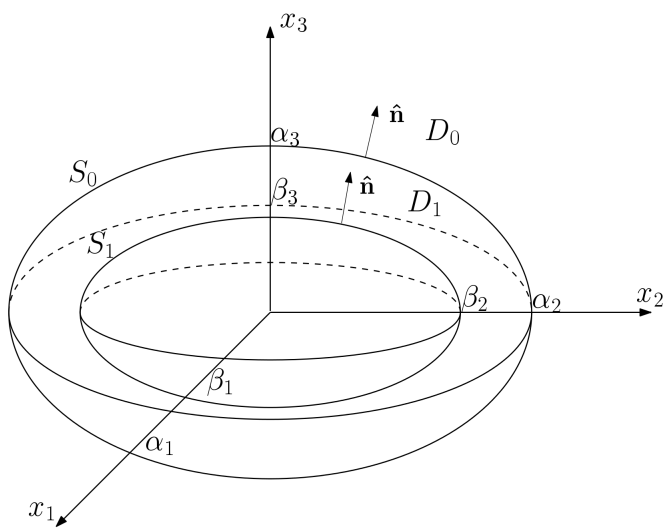


Figure 1:

The surface of the penetrable ellipsoid centered at the origin with principal semi-axes $\alpha_n$ along the axes $\mathbf{x_n}$ has the equation:

$$S_0 : \sum_{n=1}^{3} \frac{x_n^2}{\alpha_n^2} = 1 \quad , \tag{1}$$

where $\alpha_1 > \alpha_2 > \alpha_3 > 0$. The surface of the ellipsoidal acoustically soft core centered at the origin with principal semi-axes $\beta_n$ along the axes $\mathbf{x_n}$ has the equation:

$$S_1 : \sum_{n=1}^{3} \frac{x_n^2}{\beta_n^2} = 1 \quad , \tag{2}$$

where $\beta_1 > \beta_2 > \beta_3 > 0$ with $\alpha_n > \beta_n$ for $n = 1, 2, 3$. The semi-interfocal distances $h_1, h_2, h_3$ satisfy the following relations:

$$\begin{aligned} h_1^2 &= \alpha_2^2 - \alpha_3^2 = \beta_2^2 - \beta_3^2 \quad , \\ h_2^2 &= \alpha_1^2 - \alpha_3^2 = \beta_1^2 - \beta_3^2 \quad , \\ h_3^2 &= \alpha_1^2 - \alpha_2^2 = \beta_1^2 - \beta_2^2 \quad . \end{aligned} \tag{3}$$

The ellipsoidal coordinates $(\rho, \mu, \nu)$ are related to the cartesian coordinates $(x_1, x_2, x_3)$ by the relations:

$$\begin{aligned} x_1 &= \frac{h_1}{h_1 h_2 h_3} \rho\mu\nu \quad , \\ x_2 &= \frac{h_2}{h_1 h_2 h_3} \left[ \left(\rho^2 - h_3^2\right) \left(\mu^2 - h_3^2\right) \left(h_3^2 - \nu^2\right) \right]^{1/2} \quad , \\ x_3 &= \frac{h_3}{h_1 h_2 h_3} \left[ \left(\rho^2 - h_2^2\right) \left(h_2^2 - \mu^2\right) \left(h_2^2 - \nu^2\right) \right]^{1/2} \quad , \end{aligned} \tag{4}$$

where

$$-h_3 \le \nu \le h_3 \le \mu \le h_2 \le \rho < +\infty \quad . \tag{5}$$

In the ellipsoidal coordinate system, the surface $S_0$ is defined by $\rho = \alpha_1$ and the surface $S_1$ by $\rho = \beta_1$. The region $D_0$ (exterior of $S_0$) is defined by $\rho > \alpha_1$ and the region $D_1$ (between the surfaces $S_0$ and $S_1$) by $\beta_1 < \rho < \alpha_1$.
A time-harmonic acoustic plane wave

$$u^i(\mathbf{r}; \hat{\mathbf{d}}) = e^{ik_0 \hat{\mathbf{d}} \cdot \mathbf{r}} \quad , \tag{6}$$

is incident upon the layered ellipsoid. The unitary vector $\hat{\mathbf{d}}$ is the direction of propagation, $\mathbf{r}$ is the observation vector and $k_0$ is the wave number in $D_0$. The total fields $u^0$ in $D_0$ and $u^1$ in $D_1$ satisfy the Helmholtz equation:

$$\Delta u^0 + k_0^2 u^0 = 0 \quad in \quad D_0 \ , \tag{7}$$

$$\Delta u^1 + k_1^2 u^1 = 0 \quad in \quad D_1 \ . \tag{8}$$

We note that $k_1 = \eta k_0$ is the wave number in $D_1$ with $\eta$ the relative index of refraction. On the surface $S_0$ the following transmission conditions are satisfied:

$$\begin{aligned} u^0 &= u^1 \quad on \quad S_0, \\ \frac{\partial u^0}{\partial n} &= \frac{\rho_0}{\rho_1}\frac{\partial u^1}{\partial n} \quad on \quad S_0. \end{aligned} \tag{9}$$

where $\rho_j$ the mass densities in $D_j$ for $j = 0, 1$ and $\hat{\mathbf{n}}$ the outward unit normal vector on $S_0$.

Also, on the surface of the soft core the following boundary condition is satisfied:

$$u^1 = 0 \quad on \quad S_1. \tag{10}$$

The total exterior field satisfy:

$$u^0 = u^i + u^s \quad in \quad D_0, \tag{11}$$

where the scattered field $u^s$ is assumed to satisfy the Sommerfeld radiation condition:

$$\lim_{r \longrightarrow \infty} r\left(\frac{\partial u^s}{\partial r} - ik_0 u^s\right) = 0, \tag{12}$$

with $r = |\mathbf{r}|$. Let the vector $\mathbf{r}$ as well as the unitary vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{d}}$ as follows:

$$\mathbf{r} = \sum_{n=1}^{3} x_n \hat{\mathbf{x}}_{\mathbf{n}} = (x_1, x_2, x_3) \ , \ \hat{\mathbf{r}} = \sum_{n=1}^{3} o_n \hat{\mathbf{x}}_{\mathbf{n}} = (o_1, o_2, o_3) \ , \ \hat{\mathbf{d}} = \sum_{n=1}^{3} i_n \hat{\mathbf{x}}_{\mathbf{n}} = (i_1, i_2, i_3) \tag{13}$$

The low-frequency expansions of the incident and total fields are given by:

$$u^i(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}(\hat{\mathbf{d}} \cdot \mathbf{r})^n \quad , \tag{14}$$

$$u^0(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^0(\mathbf{r}) \quad in \quad D_0, \tag{15}$$

$$u^1(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} u_n^1(\mathbf{r}) \quad in \quad D_1, \tag{16}$$

where $k = k_0$. In $D_1$ the relative index of refraction $\eta$ is absorbed in the low-frequency coefficient $u_n^1$ (independent of $k$).

Replacing the fields $u^0$ and $u^1$ with their low- frequency expansions in the

Helmholtz equation and the boundary conditions that each of them respectively satisfy in our problem, we obtain the following sequence of mixed boundary value problems for the low-frequency coefficients:

$$\begin{aligned} \Delta u_n^0\left(\mathbf{r}\right) &= n(n-1)u_{n-2}^0\left(\mathbf{r}\right) \quad , \quad \mathbf{r}\in D_0 \\ \Delta u_n^1\left(\mathbf{r}\right) &= n(n-1)\eta^2 u_{n-2}^1\left(\mathbf{r}\right) \quad , \quad \mathbf{r}\in D_1 \\ u_n^0\left(\mathbf{r}\right) &= u_n^1\left(\mathbf{r}\right) \quad , \quad \mathbf{r}\in S_0 \\ \frac{\partial u_n^0\left(\mathbf{r}\right)}{\partial n} &= \frac{\rho_0}{\rho_1}\frac{\partial u_n^1\left(\mathbf{r}\right)}{\partial n} \quad , \quad \mathbf{r}\in S_0 \\ u_n^1\left(\mathbf{r}\right) &= 0 \quad , \quad \mathbf{r}\in S_1 \end{aligned} \quad , \tag{17}$$

Similarly, the low-frequency expansion of the scattered field has the form:

$$u^s(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}u_n^s(\mathbf{r}) = u_0^s(\mathbf{r}) + iku_1^s(\mathbf{r}) + O(k^2) \quad , \tag{18}$$

with $u_n^s$ the low-frequency coefficients. The first two coefficients are [6]:

$$u_0^s(\mathbf{r}) = -\frac{\zeta}{H^0}I^0(\rho) \qquad , \tag{19}$$

$$u_1^s(\mathbf{r};\hat{\mathbf{d}}) = \frac{\zeta}{H^0}\left(\frac{\zeta}{H^0}I^0(\rho)-1\right) + \sum_{n=1}^{3}\left[\frac{I_n^1(\rho)}{I_n^1(\alpha_1)}\left(\frac{I_n^1(\alpha_1)-I_n^1(\beta_1)}{H_n^1}-1\right)\right] i_n x_n \quad , \tag{20}$$

where

$$H^0 = I^0(\beta_1) + (\zeta-1)I^0(\alpha_1) \quad ,$$
$$H_n^1 = (\zeta-1)\alpha_1\alpha_2\alpha_3 I_n^1(\alpha_1)\left[I_n^1(\alpha_1)-I_n^1(\beta_1)\right] - \left[(\zeta-1)I_n^1(\alpha_1)+I_n^1(\beta_1)\right] \quad ,$$

with $\zeta = \rho_0/\rho_1$ , $I^0$ the elliptic integral of degree 0 and $I_n^1$ the elliptic integral of degree 1 and order $n = 1, 2, 3$, given by

$$I^0(\rho) = \int_\rho^\infty \frac{du}{\left[\left(u^2-h_2^2\right)\left(u^2-h_3^2\right)\right]^{1/2}},$$

$$I_n^1(\rho) = \int_\rho^\infty \frac{du}{\left(u^2-\alpha_1^2+\alpha_n^2\right)\left[\left(u^2-h_2^2\right)\left(u^2-h_3^2\right)\right]^{1/2}}.$$

The relation (20) can be written in the form:

$$u_1^s(\mathbf{r};\hat{\mathbf{d}}) = \tau + \sum_{n=1}^{3} i_n W_n x_n = \tau + \hat{\mathbf{d}}^T W\mathbf{r} \tag{21}$$

where

$$W = diag(W_n) \ , \ W_n = \frac{I_n^1(\rho)}{I_n^1(\alpha_1)} \left( \frac{I_n^1(\alpha_1) - I_n^1(\beta_1)}{H_n^1} - 1 \right),$$
$$\tau = \frac{\zeta}{H^0} \left( \frac{\zeta}{H^0} I^0(\rho) - 1 \right) \tag{22}$$

# 3 The inverse problem

For the inverse problem we will use the zeroth and first low-frequency coefficients of the scattered field $u^s$. We consider two cartesian systems with the same origin and their corresponding orthonormal bases $\{\hat{\mathbf{x}}_\mathbf{1}, \hat{\mathbf{x}}_\mathbf{2}, \hat{\mathbf{x}}_\mathbf{3}\}$ and $\{\hat{\mathbf{x}}'_\mathbf{1}, \hat{\mathbf{x}}'_\mathbf{2}, \hat{\mathbf{x}}'_\mathbf{3}\}$. The system $\{\hat{\mathbf{x}}_\mathbf{1}, \hat{\mathbf{x}}_\mathbf{2}, \hat{\mathbf{x}}_\mathbf{3}\}$ coincides with the principal directions of the unknown ellipsoids while the system $\{\hat{\mathbf{x}}'_\mathbf{1}, \hat{\mathbf{x}}'_\mathbf{2}, \hat{\mathbf{x}}'_\mathbf{3}\}$ is a known reference system. In order to specify the orientation and the size of the ellipsoid we will transform the system $\{\hat{\mathbf{x}}'_\mathbf{1}, \hat{\mathbf{x}}'_\mathbf{2}, \hat{\mathbf{x}}'_\mathbf{3}\}$ to the $\{\hat{\mathbf{x}}_\mathbf{1}, \hat{\mathbf{x}}_\mathbf{1}, \hat{\mathbf{x}}_\mathbf{3}\}$. To achieve this transformation we will use the orthogonal rotation matrix $P$ whose elements are given in terms of the Euler angles $(\alpha, \beta, \gamma)$ (Figure 2) as follows [16]:

$$P = \begin{bmatrix} cos\alpha \ cos\gamma - cos\beta \ cos\alpha \ sin\gamma & sin\alpha \ cos\gamma + cos\beta \ cos\alpha \ sin\gamma & sin\beta \ sin\gamma \\ -cos\alpha \ sin\gamma - cos\beta \ sin\alpha \ cos\gamma & -sin\alpha \ sin\gamma + cos\beta \ cos\alpha \ cos\gamma & sin\beta \ cos\gamma \\ sin\beta \ sin\alpha & -sin\beta \ cos\alpha & cos\beta \end{bmatrix} \tag{23}$$

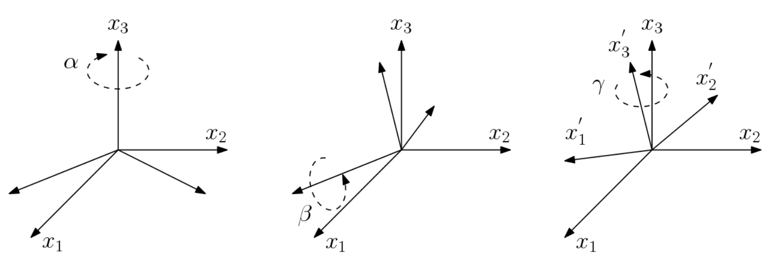


Figure 2:

Therefore vectors $\mathbf{r}, \hat{\mathbf{d}}$ satisfy the following rotation relations:

$$\begin{aligned} \mathbf{r} &= (x_1, x_2, x_3) = P\,(x_1', x_2', x_3') = P\mathbf{r}' \quad , \\ \hat{\mathbf{d}} &= (i_1, i_2, i_3) = P\,(i_1', i_2', i_3') = P\hat{\mathbf{d}}' \quad . \end{aligned} \tag{24}$$

Applying rotation (24) to (21) we obtain the following:

$$u_1^s(\mathbf{r}'; \hat{\mathbf{d}}') = \tau + \hat{\mathbf{d}}'^T P^T W P \mathbf{r}' \quad . \tag{25}$$

Our method starts by taking a point $\mathbf{r_1'} = (x_1', x_2', x_3')$ with ellipsoidal coordinates $(\rho_0, \mu_0, \nu_0)$, $\rho_0 > \alpha_1$, $x_n' \neq 0$, $n = 1, 2, 3$. Also we consider the points:

$$\mathbf{r_2'} = (x_1', -x_2', -x_3') \quad , \quad \mathbf{r_3'} = (-x_1', x_2', -x_3') \quad , \quad \mathbf{r_4'} = (-x_1', -x_2', x_3') \quad , \tag{26}$$

which belong on the surface of the ellipsoid $\rho = \rho_0$ since they are symmetric to $\mathbf{r_1'}$ over the principal axes of the reference system ([16] p. 9). We take measurements at $\mathbf{r_1'}, \mathbf{r_2'}, \mathbf{r_3'}, \mathbf{r_4'}$ for the directions of propagation:

$$\hat{\mathbf{d}}_\mathbf{1}' = \hat{\mathbf{x}}_\mathbf{1}' \quad , \quad \hat{\mathbf{d}}_\mathbf{2}' = \hat{\mathbf{x}}_\mathbf{2}' \quad , \quad \hat{\mathbf{d}}_\mathbf{3}' = \hat{\mathbf{x}}_\mathbf{3}' \quad . \tag{27}$$

Specifically, we take five measurements at $\mathbf{r_1'}$, one for the zeroth coefficient of the low-frequency expansion of the scattered field and four for the first coefficient, one for each of the directions of propagation $\hat{\mathbf{d}}_\mathbf{1}', \hat{\mathbf{d}}_\mathbf{2}', \hat{\mathbf{d}}_\mathbf{3}', -\hat{\mathbf{d}}_\mathbf{1}'$. Next we take for the first coefficient three measurements at $\mathbf{r_2'}$ for the directions of propagation $\hat{\mathbf{d}}_\mathbf{1}', \hat{\mathbf{d}}_\mathbf{2}', \hat{\mathbf{d}}_\mathbf{3}'$, two measurements at $\mathbf{r_3'}$ for $\hat{\mathbf{d}}_\mathbf{2}', \hat{\mathbf{d}}_\mathbf{3}'$ and one at $\mathbf{r_4'}$ for $\hat{\mathbf{d}}_\mathbf{3}'$. Therefore the measurements are:

$$\begin{aligned}
m_0 &= u_0^s\,(\mathbf{r_1'}) = -\frac{\zeta}{H^0} I^0(\rho_0) \ , \\
m_1 &= u_1^s(\mathbf{r_1'}; \hat{\mathbf{d}}_\mathbf{1}') = \tau + x_1'\mathbf{P_1^T} W \mathbf{P_1} + x_2'\mathbf{P_1^T} W \mathbf{P_2} + x_3'\mathbf{P_1^T} W \mathbf{P_3} \quad , \\
m_2 &= u_1^s(\mathbf{r_1'}; \hat{\mathbf{d}}_\mathbf{2}') = \tau + x_1'\mathbf{P_2^T} W \mathbf{P_1} + x_2'\mathbf{P_2^T} W \mathbf{P_2} + x_3'\mathbf{P_2^T} W \mathbf{P_3} \quad , \\
m_3 &= u_1^s(\mathbf{r_1'}; \hat{\mathbf{d}}_\mathbf{3}') = \tau + x_1'\mathbf{P_3^T} W \mathbf{P_1} + x_2'\mathbf{P_3^T} W \mathbf{P_2} + x_3'\mathbf{P_3^T} W \mathbf{P_3} \quad , \\
m_4 &= u_1^s(\mathbf{r_2'}; \hat{\mathbf{d}}_\mathbf{1}') = \tau + x_1'\mathbf{P_1^T} W \mathbf{P_1} - x_2'\mathbf{P_1^T} W \mathbf{P_2} - x_3'\mathbf{P_1^T} W \mathbf{P_3} \quad , \\
m_5 &= u_1^s(\mathbf{r_2'}; \hat{\mathbf{d}}_\mathbf{2}') = \tau + x_1'\mathbf{P_2^T} W \mathbf{P_1} - x_2'\mathbf{P_2^T} W \mathbf{P_2} - x_3'\mathbf{P_2^T} W \mathbf{P_3} \quad , \\
m_6 &= u_1^s(\mathbf{r_2'}; \hat{\mathbf{d}}_\mathbf{3}') = \tau + x_1'\mathbf{P_3^T} W \mathbf{P_1} - x_2'\mathbf{P_3^T} W \mathbf{P_2} - x_3'\mathbf{P_3^T} W \mathbf{P_3} \quad , \\
m_7 &= u_1^s(\mathbf{r_3'}; \hat{\mathbf{d}}_\mathbf{2}') = \tau - x_1'\mathbf{P_2^T} W \mathbf{P_1} + x_2'\mathbf{P_2^T} W \mathbf{P_2} - x_3'\mathbf{P_2^T} W \mathbf{P_3} \quad , \\
m_8 &= u_1^s(\mathbf{r_3'}; \hat{\mathbf{d}}_\mathbf{3}') = \tau - x_1'\mathbf{P_3^T} W \mathbf{P_1} + x_2'\mathbf{P_3^T} W \mathbf{P_2} - x_3'\mathbf{P_3^T} W \mathbf{P_3} \quad , \\
m_9 &= u_1^s(\mathbf{r_4'}; \hat{\mathbf{d}}_\mathbf{3}') = \tau - x_1'\mathbf{P_3^T} W \mathbf{P_1} - x_2'\mathbf{P_3^T} W \mathbf{P_2} + x_3'\mathbf{P_3^T} W \mathbf{P_3} \quad , \\
m_{10} &= u_1^s(\mathbf{r_1'}; -\hat{\mathbf{d}}_\mathbf{1}') = \tau - x_1'\mathbf{P_1^T} W \mathbf{P_1} - x_2'\mathbf{P_1^T} W \mathbf{P_2} - x_3'\mathbf{P_1^T} W \mathbf{P_3} \ ,
\end{aligned} \tag{28}$$

where $\mathbf{P_i} = P\mathbf{\hat{x}_i}$ is the $i^{th}$ column of matrix $P$ and $\mathbf{P_i^T} = (P\mathbf{\hat{x}_i})^T$ the $i^{th}$ row of $P^T$ for $i = 1, 2, 3$.
We note that $\tau = \tau(\rho_0) = \dfrac{m_1 + m_{10}}{2}$. Since $W$ is diagonal we obtain the following relation:

$$\mathbf{P_i^T} W \mathbf{P_j} = \mathbf{P_j^T} W \mathbf{P_i} \quad , \quad 1 \leq i, j \leq 3 \tag{29}$$

Next we construct the measurement matrix $M$ whose elements are $M_{ij} = \mathbf{P_i^T} W \mathbf{P_j}$ and from (28)-(29) can be written in terms of the measurements $m_l$ for $l = 1, ..., 10$ as follows:

$$M = \begin{bmatrix} \frac{m_1+m_4-(m_1+m_{10})}{2x_1'} & \frac{m_2+m_5-(m_1+m_{10})}{2x_1'} & \frac{m_3+m_6-(m_1+m_{10})}{2x_1'} \\ \frac{m_2+m_5-(m_1+m_{10})}{2x_1'} & \frac{m_2+m_7-(m_1+m_{10})}{2x_2'} & \frac{m_3+m_8-(m_1+m_{10})}{2x_2'} \\ \frac{m_3+m_6-(m_1+m_{10})}{2x_1'} & \frac{m_3+m_8-(m_1+m_{10})}{2x_2'} & \frac{m_3+m_9-(m_1+m_{10})}{2x_3'} \end{bmatrix} .$$

From the set of equations and matrices (28)-(29) it is concluded:

$$P^T W P = M \tag{31}$$

Since the matrix P is orthogonal:

$$W = PMP^T \quad , \tag{32}$$

which is an orthogonal similarity relation between the measurement matrix $M$ and the diagonal matrix $W$.
The measurement matrix $M$ is real and symmetric, it has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and three corresponding orthogonal eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Therefore based on the orthogonal similarity relation, we conclude that for $n = 1, 2, 3$:

$$\begin{aligned} \lambda_n &= W_n \quad , \\ \mathbf{v_n} &= (P_{n1}, P_{n2}, P_{n3}) \quad , \end{aligned} \tag{33}$$

where $P_{n1}, P_{n2}, P_{n3}$ are the elements of the $n^{th}$ row of matrix $P$. From the three orthogonal eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ we can specify the Euler angles from the elements of the rotation matrix $P$ by using the following relations [16] :

$$\alpha = sin^{-1}\left(\sqrt{1 - P_{33}^2}\right) \quad , \quad \beta = sin^{-1}\left(\frac{P_{13}}{\sqrt{1 - P_{33}^2}}\right) \quad , \quad \gamma = sin^{-1}\left(\frac{P_{31}}{\sqrt{1 - P_{33}^2}}\right) . \tag{34}$$

These angles show the orientation of the ellipsoid.
From the system of equations (32) that connect the eigenvalues $\lambda_n$ with the elements $W_n$ for $n = 1, 2, 3$ we obtain:

$$\lambda_n = \frac{I_n^1(\rho_0)}{I_n^1(\alpha_1)} \left( \frac{I_n^1(\alpha_1) - I_n^1(\beta_1)}{H_n^1} - 1 \right) . \tag{35}$$

Also we have the following relations for our system:

$$\begin{aligned} m_0 = u_0^s\left(\mathbf{r_1'}\right) &= -\frac{\zeta}{H^0} I^0(\rho_0) \quad , \\ \frac{m_1 + m_{10}}{2} = \tau(\rho_0) &= \frac{\zeta}{H^0} \left( \frac{\zeta}{H^0} I^0(\rho_0) - 1 \right) \quad . \end{aligned} \tag{36}$$

Based on relation (3) we only need to determine four out of the six semi-axes $\alpha_n, \beta_n$ and $\rho_0$ as an unknown ellipsoidal component. Therefore the system (34)-(35) contains information for the semi-axes as well as for $\rho_0$.

Summarizing this near-field method consists of the following steps:
(i) Assuming that the center of the ellipsoid is known, rotate the known reference system to the system which coincides with the principal axes of the unknown ellipsoid.
(ii) Consider an arbitrary point $\mathbf{r_1'} = (x_1', x_2', x_3')$ in the exterior region with unknown ellipsoidal coordinates $(\rho_0, \mu_0, \nu_0)$, where $\rho_0 > \alpha_1$ and $x_n' \neq 0, n = 1, 2, 3$.
(iii) Take three points $\mathbf{r_2'} = (x_1', -x_2', -x_3'), \mathbf{r_3'} = (-x_1', x_2', -x_3'), \mathbf{r_4'} = (-x_1', -x_2', x_3')$ which are symmetric to $\mathbf{r_1'}$ over the three principal axes $\mathbf{x_1'}, \mathbf{x_2'}, \mathbf{x_3'}$.
(iv) Assuming that we can isolate the coefficients $u_0^s$ and $u_1^s$ of the low-frequency expansion of the scattered field (18) take a measurement at one of these four points for an arbitrary direction of propagation for the zeroth coefficient of the scattered field.
(v) Take measurements at the four points for the first coefficient of the scattered field. In particular, take three measurements at each of the points $\mathbf{r_1'}$ and $\mathbf{r_2'}$ for directions of propagation $\hat{\mathbf{d}}_1', \hat{\mathbf{d}}_2', \hat{\mathbf{d}}_3'$ along the three principal axes of the reference system, two measurements at $\mathbf{r_3'}$ for $\hat{\mathbf{d}}_2', \hat{\mathbf{d}}_3'$ and one measurement at $\mathbf{r_4'}$ for $\hat{\mathbf{d}}_3'$. We note that we could also use other combinations of points and directions.
(vi) Take a measurement at one of these four points for a backward direction of propagation along one of the axes.

(vii) Construct a measurement matrix using the previous measurements and calculate its eigenvalues and its eigenvectors.
(viii) Use the eigenvectors to determine the orientation of the ellipsoid via the Euler angles.
(ix) Use the eigenvalues to determine the semi-axes and if it is needed the measurement of the zeroth coefficient.

# 4 A Modified Method

In the special case that the semi-focal distances $h_2, h_3$ are known, the previous method can be modified in the following one which can give additionally information for material properties contained in $\zeta$. We could also use step by step the method of section 3 but we simplify it in order to reduce the amount of measurements.
On the reference system we take an auxiliary ellipsoid of measurements big enough so that it contains the unknown ellipsoid. The surface of this ellipsoid is:

$$S_2 : \sum_{n=1}^{3} \frac{x_n'^2}{\gamma_n^2} = 1 \qquad , \tag{37}$$

where $\gamma_1 > \gamma_2 > \gamma_3$ its semi-axes and $\gamma_n > \alpha_n$ for $n = 1, 2, 3$.
We take six measurements for the first coefficient of the low-frequency expansion of the scattered field on the surface $S_2$ at the following points:

$$\mathbf{r_1'} = (\gamma_1, 0, 0) \qquad , \quad \mathbf{r_2'} = (0, \gamma_2, 0) \qquad , \quad \mathbf{r_3'} = (0, 0, \gamma_3) \qquad , \tag{38}$$

for three different directions of propagation:

$$\hat{\mathbf{d}}_\mathbf{1}' = \hat{\mathbf{x}}_\mathbf{1}' \qquad , \quad \hat{\mathbf{d}}_\mathbf{2}' = \hat{\mathbf{x}}_\mathbf{2}' \qquad , \quad \hat{\mathbf{d}}_\mathbf{3}' = \hat{\mathbf{x}}_\mathbf{3}' \qquad . \tag{39}$$

Specifically, we take three measurements at $\mathbf{r_1'}$ for directions of propagation $\hat{\mathbf{d}}_\mathbf{1}', \hat{\mathbf{d}}_\mathbf{2}', \hat{\mathbf{d}}_\mathbf{3}'$, two measurements at $\mathbf{r_2'}$ for directions of propagation $\hat{\mathbf{d}}_\mathbf{2}', \hat{\mathbf{d}}_\mathbf{3}'$ and one measurement at $\mathbf{r_3'}$ for direction of propagation $\hat{\mathbf{d}}_\mathbf{3}'$.
Also, we take a measurement at $\mathbf{r_1'}$ for the zeroth coefficient as well a measurement at $\mathbf{r_1'}$ for direction of propagation $-\hat{\mathbf{d}}_\mathbf{1}'$ for the first coefficient. Since the points $\mathbf{r_1'}, \mathbf{r_2'}, \mathbf{r_3'}$ are on surface $S_2$, their corresponding ellipsoidal coordinates will all have $\rho = \gamma_1$ which can also be verified from (4). Therefore the

measurements will be:

$$
\begin{aligned}
m_0 &= u_0^s\left(\mathbf{r_1'}\right) = -\frac{\zeta}{H^0} I^0(\gamma_1) \quad , \\
m_1 &= u_1^s(\mathbf{r_1'};\hat{\mathbf{d}}_\mathbf{1}') = \tau + \gamma_1 \mathbf{P_1^T} W \mathbf{P_1} \quad , \\
m_2 &= u_1^s(\mathbf{r_2'};\hat{\mathbf{d}}_\mathbf{2}') = \tau + \gamma_2 \mathbf{P_2^T} W \mathbf{P_2} \quad , \\
m_3 &= u_1^s(\mathbf{r_3'};\hat{\mathbf{d}}_\mathbf{3}') = \tau + \gamma_3 \mathbf{P_3^T} W \mathbf{P_3} \quad , \\
m_4 &= u_1^s(\mathbf{r_1'};\hat{\mathbf{d}}_\mathbf{2}') = \tau + \gamma_1 \mathbf{P_2^T} W \mathbf{P_1} \quad , \\
m_5 &= u_1^s(\mathbf{r_1'};\hat{\mathbf{d}}_\mathbf{3}') = \tau + \gamma_1 \mathbf{P_3^T} W \mathbf{P_1} \quad , \\
m_6 &= u_1^s(\mathbf{r_2'};\hat{\mathbf{d}}_\mathbf{3}') = \tau + \gamma_2 \mathbf{P_3^T} W \mathbf{P_2} \quad , \\
m_7 &= u_1^s(\mathbf{r_1'};-\hat{\mathbf{d}}_\mathbf{1}') = \tau - \gamma_1 \mathbf{P_1^T} W \mathbf{P_1} \quad .
\end{aligned}
\tag{40}
$$

We note that $\tau = \tau(\gamma_1) = \dfrac{m_1 + m_7}{2}$. Next we construct the measurement matrix $M$ whose elements are $M_{ij} = \mathbf{P_i^T} W \mathbf{P_j}$ and can be written in terms of the measurements $m_l$ for $l = 1, ..., 7$ as follows:

$$
M = \begin{bmatrix}
\frac{m_1 - m_7}{2\gamma_1} & \frac{2m_4 - (m_1 + m_7)}{2\gamma_1} & \frac{2m_5 - (m_1 + m_7)}{2\gamma_1} \\
\frac{2m_4 - (m_1 + m_7)}{2\gamma_1} & \frac{2m_2 - (m_1 + m_7)}{2\gamma_2} & \frac{2m_6 - (m_1 + m_7)}{2\gamma_2} \\
\frac{2m_5 - (m_1 + m_7)}{2\gamma_1} & \frac{2m_6 - (m_1 + m_7)}{2\gamma_2} & \frac{2m_3 - (m_1 + m_7)}{2\gamma_3}
\end{bmatrix} .
$$

We note here that we can avoid taking the measurement $m_7$ since $\tau = \dfrac{m_0}{I^0(\gamma_1)}(m_0 + 1)$ and the corresponding measurement matrix is:

$$
M = \begin{bmatrix}
\dfrac{m_1 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_1 I^0(\gamma_1)} & \dfrac{m_4 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_1 I^0(\gamma_1)} & \dfrac{m_5 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_1 I^0(\gamma_1)} \\
\dfrac{m_4 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_1 I^0(\gamma_1)} & \dfrac{m_2 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_2 I^0(\gamma_1)} & \dfrac{m_6 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_2 I^0(\gamma_1)} \\
\dfrac{m_5 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_1 I^0(\gamma_1)} & \dfrac{m_6 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_2 I^0(\gamma_1)} & \dfrac{m_3 I^0(\gamma_1) - m_0(m_0+1)}{\gamma_3 I^0(\gamma_1)}
\end{bmatrix} .
$$

The measurement matrix $M$ can also be constructed from other combinations of points and directions.

From the previous set of equations and matrices it is concluded:

$$
P^T W P = M \tag{43}
$$

Since the matrix P is orthogonal:

$$
W = P M P^T \quad , \tag{44}
$$

which is an orthogonal similarity relation between the measurement matrix $M$ and the diagonal matrix $W$.

The measurement matrix $M$ is real and symmetric, it has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and three corresponding orthogonal eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Therefore based on the orthogonal similarity relation, we have the system of equations (32). From the three orthogonal eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ we can specify the Euler angles as elements of the rotation matrix $P$ by using the relations (33). These angles show the orientation of the ellipsoid.

From the system of equations that connect the eigenvalues $\lambda_n$ with the elements $W_n$ for $n = 1, 2, 3$ we obtain:

$$\lambda_n = \frac{I_n^1(\gamma_1)}{I_n^1(\alpha_1)} \left( \frac{I_n^1(\alpha_1) - I_n^1(\beta_1)}{H_n^1} - 1 \right) . \tag{45}$$

Based on (3) we only need to determine one of the semi-axes $\alpha_n$ and one of the semi-axes $\beta_n, \quad n = 1, 2, 3$. From system of equations (42) we can obtain information regarding these semi-axes as well as the material properties contained in $\zeta$.

To calculate the elliptic integrals $I^0(\gamma_1), I_n^1(\gamma_1)$ for the known $\gamma_1$ we use the following relations connecting them to the incomplete elliptic integrals of the first $(F(\phi, \alpha))$ and second kind $(E(\phi, \alpha))$ [1]:

$$\begin{aligned}
I_0^1(\gamma_1) &= \frac{1}{h_2} F(\phi, \alpha) \qquad , \\
I_1^1(\gamma_1) &= \frac{1}{h_2 h_3^2} \left( F(\phi, \alpha) - E(\phi, \alpha) \right) \qquad , \\
I_2^1(\gamma_1) &= \frac{h_2}{h_1^2 h_3^2} E(\phi, \alpha) - \frac{1}{h_2 h_3^2} F(\phi, \alpha) - \frac{1}{h_1^2} \frac{\sqrt{\gamma_1^2 - h_2^2}}{\gamma_1 \sqrt{\gamma_1^2 - h_3^2}} \qquad , \\
I_3^1(\gamma_1) &= -\frac{1}{h_1^2 h_2} E(\phi, \alpha) + \frac{1}{h_1^2} \frac{\sqrt{\gamma_1^2 - h_3^2}}{\gamma_1 \sqrt{\gamma_1^2 - h_2^2}} \qquad ,
\end{aligned} \tag{46}$$

where $\phi = sin^{-1} \dfrac{h_2}{\gamma_1}$ the amplitude and $\alpha = sin^{-1} \dfrac{h_3}{h_2}$ the modular angle.

The measurements $m_0$ and $m_7$ were only used for the calculation of $\tau$, otherwise the system would be much more complicated. We note that in case we know the orientation and the material properties of the unknown ellipsoid then we only need to take two measurements, one for the low-frequency coefficient $u_0^s$ and one for $u_1^s$, at an arbitrary point of the surface $S_2$ for an arbitrary direction of propagation, in order to determine the size of the penetrable ellipsoid and its acoustically soft confocal ellipsoidal core.

# 5 Physically Degenerate form

If the regions $D_0$ and $D_1$ have the same mass densities $\rho_0 = \rho_1$ and the same mean compresibilities we have no scattering on the surface $S_0$ and the problem is reduced to the acoustically soft triaxial ellipsoid of surface $S_1$ and semi-axes $\beta_1 > \beta_2 > \beta_3 > 0$ [15],[6]. Replacing $\zeta = 1$ to (19)-(22), we obtain the following:

$$u_0^s(\mathbf{r}) = -\frac{I^0(\rho)}{I^0(\beta_1)} \qquad , \qquad u_1^s(\mathbf{r}) = \tau + \mathbf{d}^{\mathbf{T}} W \mathbf{r} \qquad , \tag{47}$$

where

$$W = diag(W_n) \quad , \quad W_n = -\frac{I_n^1(\rho)}{I_n^1(\beta_1)} \quad , \quad \tau = -\frac{1}{I^0(\beta_1)}\left(\frac{I^0(\rho)}{I^0(\beta_1)} - 1\right) \quad , \tag{48}$$

Applying step by step the method formulated in section 3 for this case, we construct the measurement matrix $M$. The eigenvectors will determine the Euler angles and therefore the orientation of the ellipsoid and the eigenvalues will give information about the semi-axes. Specifically based on (32), the system of equations which gives information about the semi-axes is:

$$\begin{aligned} \lambda_n &= -\frac{I_n^1(\rho_0)}{I_n^1(\beta_1)} \quad n = 1, 2, 3 \quad , \\ m_0 &= -\frac{I^0(\rho_0)}{I^0(\beta_1)} \quad . \end{aligned} \tag{49}$$

These results cover the near-field solution that corresponds to the inverse acoustic scattering problem for the soft ellipsoid. [7] For this case the backward measurement is only used for the construction of the measurement matrix.

# 6 Geometrically Degenerate forms

The sphere is a special case of the ellipsoid when $\alpha_1 = \alpha_2 = \alpha_3 = R_1 > \beta_1 = \beta_2 = \beta_3 = R_2$, with $R_1$ the radius of the penetrable sphere and $R_2$ the radius of the soft concentric spherical core. Then $h_1 = h_2 = h_3 = \mu = \nu = 0$. Thus, for this case the elliptic integrals are [6]

$$I^0(\rho) = \frac{1}{\rho} \qquad , \qquad I_n^1(\rho) = \frac{1}{3\rho^3} \quad . \tag{50}$$

The zeroth and first coefficients of the low-frequency expansion of the scattered field are:

$$
\begin{aligned}
u_0^s(\mathbf{r};\hat{\mathbf{d}}) &= -\frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\frac{1}{r} \quad , \\
u_1^s(\mathbf{r};\hat{\mathbf{d}}) &= \frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\left[\frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\frac{1}{r}-1\right] \\
&\quad + \sum_{n=1}^{3}\frac{R_1^3}{r^3}\left[\frac{(R_2^3-R_1^3)}{\frac{1}{3}(\zeta-1)(R_2^3-R_1^3)-[R_2^3(\zeta-1)+R_1^3]}-1\right] i_n x_n \quad .
\end{aligned}
\tag{51}
$$

The first low-frequency coefficient of the scattered field takes the following form:

$$
u_1^s(\mathbf{r};\hat{\mathbf{d}}) = \tau_s + \hat{\mathbf{d}}^T I w_s \mathbf{r} = \tau_s + w_s \hat{\mathbf{d}}^T \mathbf{r} \quad . \tag{52}
$$

Applying (24) to (29) we obtain:

$$
u_1^s(\mathbf{r}';\hat{\mathbf{d}}') = \tau_s + w_s \hat{\mathbf{d}}'^T P^T P \mathbf{r}' = \tau_s + w_s \hat{\mathbf{d}}'^T \mathbf{r}' \quad , \tag{53}
$$

with

$$
\begin{aligned}
w_s &= \frac{R_1^3}{r^3}\left[\frac{(R_2^3-R_1^3)}{\frac{1}{3}(\zeta-1)(R_2^3-R_1^3)-[R_2^3(\zeta-1)+R_1^3]}-1\right] \quad , \\
\tau_s &= \frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\left[\frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\frac{1}{r}-1\right] \quad .
\end{aligned}
\tag{54}
$$

Because of the symmetry of the sphere we do not need to determine the rotation matrix $P$ and the previous method degenerates into a much simpler method. In particular, we only need to determine the size of the sphere and its spherical core, specifying the radii $R_1, R_2$. We will take a measurement at a point $\mathbf{r}_1' = R\hat{\mathbf{r}}_1'$ of the reference system for direction of propagation $\hat{\mathbf{d}}_1'$ for $\hat{\mathbf{r}}_1' = \hat{\mathbf{d}}_1'$ and $R > R_1 > R_2$ .
Therefore the measurements will be:

$$
\begin{aligned}
m_0 &= u_0^s(\mathbf{r}_1') = -\frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\frac{1}{R} \quad , \\
m_1 &= u_1^s(\mathbf{r}_1';\hat{\mathbf{d}}_1') = \tau_s + w_s R \quad .
\end{aligned}
\tag{55}
$$

The system which gives information about $R_1, R_2$ is the following:

$$
\begin{aligned}
m_0 = \; & -\frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\frac{1}{R} \quad , \\
m_1 = \; & \frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\left[\frac{\zeta R_1 R_2}{[R_1+(\zeta-1)R_2]}\frac{1}{R}-1\right] \\
& + \frac{R_1^3}{R^2}\left[\frac{(R_2^3-R_1^3)}{\frac{1}{3}(\zeta-1)(R_2^3-R_1^3)-[R_2^3(\zeta-1)+R_1^3]}-1\right] \quad .
\end{aligned}
\tag{56}
$$

The needle is a special case of the ellipsoid when $\alpha_1 >> \alpha_2 = \alpha_3$ and $\beta_1 >> \beta_2 = \beta_3$.
For this case the elliptic integrals are [6]

$$
\begin{aligned}
& I^0(\rho) = \frac{1}{2h_3}\ln\left(\frac{\rho+h_3}{\rho-h_3}\right) \quad , \\
& I_1^1(\rho) = \frac{1}{h_3^2}\left(I^0(\rho)-\frac{1}{\rho}\right) \quad , \quad I_2^1(\rho) = I_3^1(\rho) = -\frac{1}{2h_3^2}\left(I^0(\rho)-\frac{\rho}{\rho^2-{h_3}^2}\right) \quad .
\end{aligned}
\tag{57}
$$

Also,

$$
I^0(\alpha_1) \sim \frac{\ln(2\sigma_\alpha)}{\alpha_1} \quad \text{for } \sigma_\alpha \to \infty \quad , \quad I^0(\beta_1) \sim \frac{\ln(2\sigma_\beta)}{\beta_1} \quad \text{for } \sigma_\beta \to \infty \quad ,
\tag{58}
$$

where $\sigma_a = \alpha_1/\alpha_2$ and $\sigma_\beta = \beta_1/\beta_2$.
Similarly with the case of the ellipsoid we construct the measurement matrix whose eigenvalues and eigenvectors can determine the sizze and the orientation of the ellipsoid.

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