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Scheduling a Three-machine Flow-shop Problem with a Single Server and Equal Processing Times

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Abstract

We consider the problem of three-machine flow-shop scheduling with a single server and equal processing times, we show that this problem is NP-hard in the strong sense and present an improved Y-H algorithm for it with worst-case bound $4/3$.

Mathematics Subject Classification : 90B35

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1 Introduction

In the three-machine flow-shop scheduling problem we study, the input instance consists of n jobs with a single server and equal processing times. Each job J_j

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requires three operations $O_{1j}, O_{2,j}$ and $O_{3j} (j = 1,2,...,n)$, which are performed on machine M_1, M_2 and M_3, respectively. The processing times of job J_j on machine M_i, i.e., the duration of operation $O_{i,j}$, is $p_{i,j} (i = 1,2,3)$. In this paper we will focus on equal processing times, that is $p_{i,j} = p$. For each job, the second operation cannot be started before the first operation is completed. A setup times $s_{i,j}$ is needed before the first job is processed on machine M_i. Each setup operation must be performed by the server, which can only perform one operation at a time. The objective is to compute a non-preemptive schedule of those jobs on m machines that minimize makespan. In the standard scheduling notation [2], the problem can be described as the $F3, S1|p_{ij} = p|C_{\max}$ problem.

It is well known, S.M. Johnson [4], the $F3|\,|C_{\max}$ problem has a maximal polynomial solvable. P. Brucker [1] show that the $F2, S1|p_{ij} = p|C_{\max}$ problem is NP-hard in the ordinary sense. In this paper, we will show that the $F3, S1|p_{ij} = p|C_{\max}$ problem is NP-hard in the strong sense.

The remainder of this paper is organized as follows. In section 2, we will discuss the complexity of the $F3, S1|p_{ij} = p|C_{\max}$ problem and prove that this problem is NP-hard in the strong sense. In section 3, we will present an improved Y-H [5] algorithm and shown that the worst-case is $4/3$, the bound is tight.

2 Complexity of the $F3, S1|p_{ij} = p|C_{\max}$ problem

In this section, we consider problem in which we have three machines M_1, M_2, M_3 a single server M_s and n jobs J_j with processing times $p_{1,j}, p_{2,j}, p_{3,j}$ and server times $s_{1,j}, s_{2,j}, s_{3,j}$ on machine M_1, M_2 and M_3, respectively.

Lemma 2.1 [6] Consider the $F3, S1|p_{ij} = p|C_{\max}$ problem with processing times $p_{i,j}$ and server times $s_{i,j}$, where $i = 1,2,3$ and $j = 1,2,...,n.$ Then

$$C(\sigma,\tau) = \max_{1\le k\le n}\{\sum_{i\le\sigma^{-1}(k)}(s_{1,\sigma(i)} + p_{1,\sigma(i)}) + \sum_{i\le\tau^{-1}(k)}^{\pi^{-1}(j)}(s_{2,\tau(i)} + p_{2,\tau(i)}) + \sum_{l\ge\pi^{-1}(j)}(s_{3,\pi(l)} + p_{3,\pi(l)})\} \quad (2.1)$$

where $\sigma^{-1}(k), \tau^{-1}(k)$ and $\pi^{-1}(j)$ denote the positions of job k in sequence σ, τ, π, respectively.

Theorem 2.1 The $F3, S1|p_{ij} = p|C_{\max}$ problem is NP-hard in the strong sense.

Proof. We prove the $F3, S1|p_{ij} = p|C_{\max}$ problem is NP -hard in the strong sense through a reduction from the $3-Partition$ problem [3], which is known to be NP -hard in the strong sense, to the $F3, S1|p_{ij} = p|C_{\max}$ problem. The $3-Partition$ problem is then stated as:

$3-Partition$**:** Given a set of positive integers $X = \{x_1, x_2,..., x_{3r}\}$, and a positive integer b with:

$$\sum_{j=1}^{3r} x_j = rb, \quad b/4 < x_j < b/2, \ \forall j = 1,2,...,r \quad (2.2)$$

Decide whether there exists a partition of X into r disjoint 3-element subset

$$\{X_1, X_2,..., X_r\} \text{ such that } i = 1,2,...,r \quad (2.3)$$

Given any instance of the $3-Partition$ problem, we define the following instance of the $F3, S1|p_{ij} = p|C_{\max}$ problem with four types of jobs:

(1) P -job: $s_{1,j} = x_j,\ p_{1,j} = b,\ s_{2,j} = 0,\ p_{2,j} = b,\ s_{3,j} = 0,\ p_{3,j} = b \quad (j = 1,2,...,3r)$

(2) U -job: $s_{1,j} = 0,\ p_{1,j} = b,\ s_{2,j} = 2b,\ p_{2,j} = b,\ s_{3,j} = 2b,\ p_{3,j} = b \quad (j = 1,2,...,r)$

(3) V -job: $s_{1,j} = b,\ p_{1,j} = b,\ s_{2,j} = 0,\ p_{2,j} = b,\ s_{3,j} = 0,\ p_{3,j} = b \quad (j = 1,2,...,r)$

(4) W -job: $s_{1,j} = 0,\ p_{1,j} = b,\ s_{2,j} = 0,\ p_{2,j} = b,\ s_{3,j} = 0,\ p_{3,j} = b \quad (j = 1,2,...,r)$

The threshold $y = 4br + 10b$ and the corresponding decision problem is: Is there a schedule S with makespan $C(S)$ not greater than $y = 4br + 10b$?

Observe that all processing times are equal to b .To prove the theorem we show that in this constructed if the $F3, S1 \left| p_{ij} = p \right| C_{\max}$ problem a schedule S_0 satisfying

$$C_{\max}(S_0) \le y = 4br + 10b$$

exists if and only if the $3-Partition$ problem has a solution.

Suppose that the $3-Partition$ problem has a solution, and $X_j (j = 1,2,...,r)$ are the required subsets of set X . Notice that each set X_j contains precisely elements, since

$$b/4 < x_j < b/2,$$

and

$$\sum_{j=1}^{3m} x_j = rb, \quad \text{for all} \quad j = 1,2,...,r.$$

Let σ denote a sequence of the elements of set X for which

$$X_j = \{\sigma(3j-2), \sigma(3j-1), \sigma(3j)\},$$

for $j = 1,2,...,r$.

The desired schedule S_0 exists and can be described as follows. No machine has intermediate idle time. Machine M_1 process the P -jobs, U -jobs, V -jobs, and W -jobs in order of the sequence σ , i.e., in the sequence

$$\sigma = (P_{1,1}, P_{1,2}, P_{1,3}, U_{1,1}, V_{1,1}, W_{1,1}, ..., P_{1,3r-2}, P_{1,3r-1}, P_{1,3r}, U_{1,r}, V_{1,r}, W_{1,r})$$

While machine M_2 process the P -jobs, U -jobs, V -jobs, and W -jobs in the order of sequence τ , i.e., in the sequence

$$\tau = (U_{2,1}, P_{2,1}, P_{2,2}, P_{2,3}, V_{2,1}, W_{2,1}, ..., U_{2,r}, P_{2,3r-2}, P_{2,3r-1}, P_{2,3r}, V_{2,r}, W_{2,r})$$

machine M_3 process the P -jobs, U -jobs, V -jobs, and W -jobs in the order of sequence π , i.e., in the sequence

$$\pi = (U_{3,1}, P_{3,1}, P_{3,2}, P_{3,3}, V_{3,1}, W_{3,1}, \ldots, U_{3,r}, P_{3,3r-2}, P_{3,3r-1}, P_{3,3r}, V_{3,r}, W_{3,r})$$

as indicated in Figure 1.

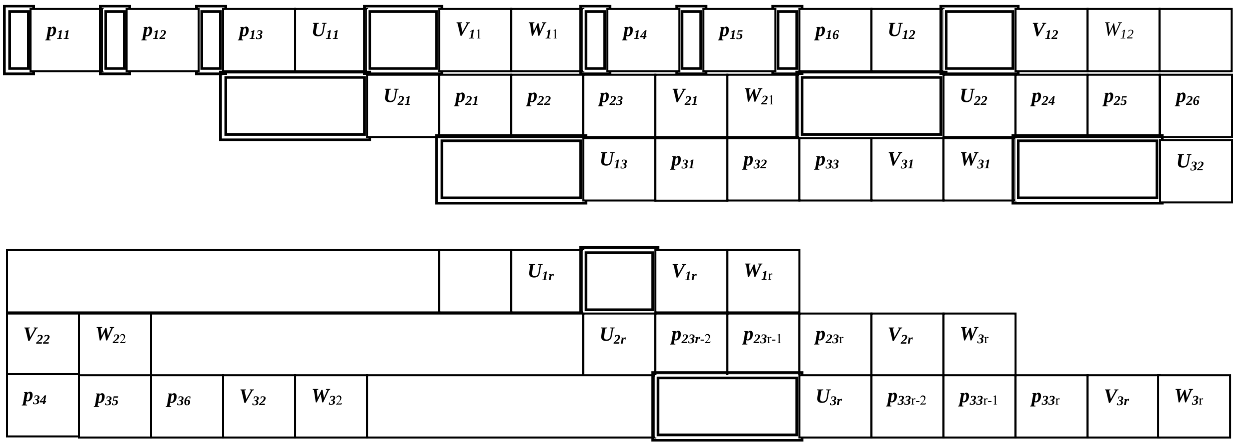

Figure 1: Gantt chart for the $F3, S1 | p_{ij} = p | C_{\max}$ problem

Then we define the sequence σ, τ and π shown in Figure 1. Obviously, these sequence σ, τ and π fulfills $C(\sigma, \tau, \pi) \le y$.

Conversely, assume that the flow-shop scheduling problem has a solution σ, τ and π with $C(\sigma, \tau, \pi) \le y$.

By setting

$$\sigma(j) = j (j = 1,2,3), \tau(j) = 1, \pi(j) = 1$$

in (2.1), we get for all sequence σ, τ and π:

$$C(\sigma, \tau, \pi) \ge (s_{1,1} + p_{1,1} + s_{1,2} + p_{1,2} + s_{1,3} + p_{1,3})$$
$$+ U_{1,1} + U_{2,1} + \sum_{\lambda=1}^{n} (s_{3,\pi_\lambda} + p_{3,\pi_\lambda}) = 4rb + 10b = y.$$

Thus, for the sequence σ, τ and π with

$$C(\sigma, \tau, \pi) = y.$$

We may conclude that:

(1) machine M_1 process jobs in the interval $[0,4rb+4b]$, without idle times,

(2) machine M_2 process jobs in the interval $[3b,4rb+7b]$, without idle times,

(3) machine M_3 process jobs in the interval $[6b,4rb+10b]$, without idle times,

(4) server S process jobs in the interval $[0,4rb+4b]$, without idle times.

Now, we will prove that the

$$\sum_{i\in X_1}(s_{1,i}+p_{1,i})=4b.$$

If $\sum_{i\in X_1}(s_{1,i}+p_{1,i})\geq 4b$, then U_{21}-job cannot start processing at time $4b$, which contradicts (2). If $\sum_{i\in X_1}(s_{1,i}+p_{1,i})\leq 4b$, then there is idle time before machine M_1 process job $U_{1,1}$, which contradicts (1). Thus, we have

$$\sum_{i\in X_1}(s_{1,i}+p_{1,i})=4b.$$

Since $p_{1,1}=p_{1,2}=p_{1,3}=b, s_{1,i}=x_i$, then

$$\sum_{i\in X_1}(s_{1,i}+p_{1,i})=(s_{1,1}+p_{1,1}+s_{1,2}+p_{1,2}+s_{1,3}+p_{1,3})=3b+\sum_{i\in X_1}x_i=4b$$

$$\sum_{i\in X_1}x_i=b$$

The set X_1 give a solution to the $3-Partition$ problem.

Analogously, we show that the remaining sets $X_2,X_3,...,X_r$ separated by the jobs $1,2,...,r$ contain 3-element and fulfill

$$\sum_{j\in X_j}x_j=b,\quad \text{for } j=1,2,...,r.$$

Thus, $X_1,X_2,...,X_r$ define a solution of the $3-Partition$ problem. □

3 Algorithm for the $F3,S1|p_{ij}=p|C_{\max}$ problem

For the $F3,S1|p_{ij}=p|C_{\max}$ problem, we consider an improved Y - H simple algorithm.

Algorithm 1

Step1 If

$$\min\{s_{1,i}+p_{1,i},s_{2,j}+p_{2,j}\}\le\min\{s_{1,j}+p_{1,j},s_{2,i}+p_{2,i}\}$$

$$\min\{s_{1,i}+p_{1,i},s_{3,j}+p_{3,j}\}\le\min\{s_{1,j}+p_{1,j}.s_{3,i}+p_{3,i}\}$$

$$\min\{s_{2,i}+p_{2,i},s_{3,j}+p_{3,j}\}\le\min\{s_{2,j}+p_{2,j},s_{3,i}+p_{3,i}\}$$

Arrange job J_i before job J_j.

Step2 Repeat step1 until all jobs are scheduled.

Theorem 3.2 The $F3,S1|p_{ij}=p|C_{\max}$ problem, let S_0 be a schedule created by Algorithm 1, S^* be the optimal solution for the $F3,S1|p_{ij}=p|C_{\max}$ problem**,** then

$$C_{\max}(S^0)/C_{\max}(S^*)\le 4/3 .$$

The bound is tight.

Proof. For a schedule S, let $I_i(S)(i=1,2,3)$ denote the total idle times on machine M_i.

Considering the path composed of machine M_1 operations of jobs $1,2,...,r$, machine M_2 operation of job r, and machine M_3 operation of job r, we obtain that

$$C_{\max}(S^0)=\sum_{i=1}^{r}(s_{1,i}+p_{1,i})+I_1(S^0)+s_{2,r}+p_{2,r}+s_{3,r}+p_{3,r}$$

Considering the path composed of machine M_1 operation of job1, machine M_2

operations of jobs $1,2,...,r$, and machine M_3 operation of job r, we obtain that

$$C_{\max}(S^0) = s_{1,1} + p_{1,1} + \sum_{i=1}^{r}(s_{2,i} + p_{2,i}) + I_2(S^0) + s_{3,r} + p_{3,r}$$

Considering the path composed of machine M_1 operation of job1, machine M_2 operation of job1 and machine M_3 operations of jobs $1,2,...,r$, we obtain that

$$C_{\max}(S^0) = s_{1,1} + p_{1,1} + s_{2,1} + p_{2,1} + \sum_{i=1}^{r}(s_{3,i} + p_{3,i}) + I_3(S^0)$$

$$\begin{aligned}
3C_{\max}(S^0) &= \sum_{i=1}^{r}(s_{1,i} + p_{1,i}) + I_1(S^0) + s_{2,r} + p_{2,r} + s_{3,r} + p_{3,r} + s_{1,1} + p_{1,1} + \sum_{i=1}^{r}(s_{2,i} + p_{2,i}) \\
&\quad + I_2(S^0) + s_{3,r} + p_{3,r} + s_{1,1} + p_{1,1} + s_{2,1} + p_{2,1} + \sum_{i=1}^{r}(s_{3,i} + p_{3,i}) + I_3(S^0) \\
&= (\sum_{i=1}^{r}(s_{1,i} + p_{1,i}) + I_1(S^0)) + (\sum_{i=1}^{r}(s_{2,i} + p_{2,i}) + I_2(S^0) + (\sum_{i=1}^{r}(s_{3,i} + p_{3,i}) + I_3(S^0)) \\
&\quad + (s_{1,1} + p_{1,1} + s_{1,1} + p_{1,1} + s_{2,1} + p_{2,1} + s_{2,r} + p_{2,r} + s_{3,r} + p_{3,r}) \\
&\le 4C_{\max}(S^*)
\end{aligned}$$

$$C_{\max}(S^0)/C_{\max}(S^*) \le 4/3.$$

To prove the bound is tight, introduce the following example as show in Figure 2 and Figure 3.

(1) $s_{1,1} = 0,\ p_{1,1} = 1,\ s_{2,1} = 1,\ p_{1,2} = 1,\ s_{3,1} = 1,\ p_{1,3} = 1,$

(2) $s_{1,2} = 0,\ p_{2,1} = 1,\ s_{2,2} = 0,\ p_{2,2} = 1,\ s_{3,2} = 0,\ p_{3,2} = 1,$

(3) $s_{1,3} = 1,\ p_{1,3} = 1,\ s_{2,3} = 1,\ p_{2,3} = 1,\ s_{3,3} = 1,\ p_{3,3} = 1.$

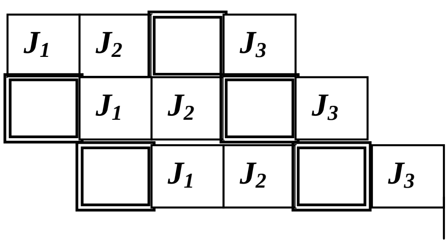

Figure 2: $C_{\max}(S^*)$ $C_{\max}(S^*) = 6$

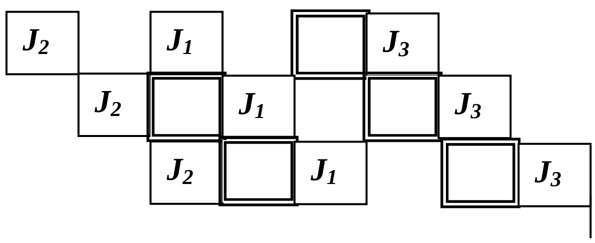

Figure 3: $C_{\max}(S^0)$ $C_{\max}(S^0) = 8$

So we have

$$C_{\max}(S^0)/C_{\max}(S^*) = 8/6 = 4/3,$$

the bound is tight. □

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