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The Two Pricing Comparisons of Mortgage Common Insurance

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Abstract

Taking advantage of martingale pricing method and insurance actuary pricing method,we obtain the martingale pricing formula and the insurance actuary pricing formula of the mortgage common insurance , when the real estate price is driven by general O-U process; Furthermore, we prove that the two pricing results is different and the insurance actuary pricing is essentially a arbitrage pricing.

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1 Introduction

Housing mortgage guarantee insurance refers to the insurance which credit bank requires the housing buyers who lending money from the bank to cover, and who should paying a certain amount of premium to insurance company when they lending money from the bank . Insurance company as a guarantee to reimburse the bank for the loan of housing buyers, the bank properly lending money to housing buyers and give a favor to borrowers on the interest and loan deadline and so on.

If borrowers break the contract, within the loan deadline, who can't reimburse on time .Thus, the underwriter (the insurance company) should compensate the loss of the bank. There are two types of housing mortgage guarantee insurance, the one is all secured and the other is partially secured. Partially secured refers to the insurance company guaranteed a certain amount of housing mortgage balance in order to reduce the liability of credit risk [1-3].

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This article, on the basis of foreign insurance experience in mortgage insurance, conducts some innovative design about the mortgage guarantee insurance. If the loss is within the proportion of guarantee k_1 it is entirely borne by the insurance company; if it is beyond k_1, it will be allocated between the insurance companies and lending institutions in accordance with the proportion k_2, which can be called co-insurance. Suppose A_0 (loan principal and interest) is the amount of the guarantee, $M(T)$ is the unpaid amount for the moment T, $H(T)$ is the value of the property for the moment T and α is the housing value ratio (constant) after realizing the mortgage, the income at the expiration of the joint insurance policies hold by lending institutions can be expressed as:

$$V_T = \begin{cases} \max(M(T)-\alpha H(T),0), & \max(M(T)-\alpha H(T),0) < k_1 A_0, \\ k_1 A_0 + k_2[M(T)-\alpha H(T) - k_1 A_0], & M(T)-\alpha H(T) \ge k_1 A_0. \end{cases} \tag{1}$$

2 The Construction of Mathematical Models

Given the financial market in continuous time, taking 0 as now and T as the due date; Given a complete probability space (Ω, F, P), assume that the unpaid amount $M(T)$ is a constant at moment T (can be obtained by credit evaluation of risk and suppose $M(T) > k_1 A_0$), and the risk-free rate $r(t)$ is the function of time t, property values $H(t)$ meet stochastic differential equation as follows:

$$\frac{\mathrm{d}H(t)}{H(t)} = [\mu(t) - a\ln(H(t))]\mathrm{d}t + \sigma(t)\mathrm{d}\widetilde{B}(t), \quad H(0) = H \tag{2}$$

Where, $\sigma(t)$ are continuous functions of the time t, $\sigma(t) > 0, \{B(t)\}_{0\le t\le T}$ is one-dimensional standard Brownian Motion of (Ω, F, P), $(F_t)_{0\le t\le T}$ is the corresponding natural information flow, $F_t = F$. The role of the constant $a(>0)$ is that when prices rise to a certain height, it makes a downward trend in $H(t)$, and the expected rate of return in this model depends on the property values.

Lemma 2.1 Assume property values meet (2), then we have

$$H(t) = H^{e^{-at}} \exp\{\int_0^t [\mu(s) - \frac{1}{2}\sigma^2(s)]e^{-as}\mathrm{d}s + e^{-at}\int_0^t e^{as}\sigma(s)\mathrm{d}B(s)\} \tag{3}$$

$$\mathrm{E}[H(t)] = H^{e^{-\alpha t}} \exp\{\int_0^t [\mu(s) - \frac{1}{2}\sigma^2(s)]e^{-as}\mathrm{d}s + \frac{1}{2}\int_0^t \sigma^2(t)e^{-2as}\mathrm{d}s\} \tag{4}$$

3 The Martingale Pricing of Housing Mortgage Common Insurance

Here,we assume the financial market is complete and no arbitrage.The traditional martingale pricing methods is used to obtain the insurance pricing.

Let $\theta(t) = \frac{\mu(t) - a\ln(H(t)) - r(t)}{\sigma(t)}, 0 \le t \le T$, be a process adapted to $(F_t)_{0\le t\le T}$. Define a new probability $\widetilde{\mathrm{P}}$ by:

$$\frac{\mathrm{d}\widetilde{\mathrm{P}}}{\mathrm{dP}}=\exp\{-\int_0^T\theta(u)\mathrm{d}B(u)-\frac{1}{2}\int_0^T(\theta(u))^2\mathrm{du}\},$$

and define a process $\{\tilde{B}(t)\}_{0\le t\le T}$ by $\mathrm{d}\tilde{B}(t)$ □ $\theta(t)\mathrm{d}t+\mathrm{d}B(t)$.

According to the Girsanov Theorem,the probability P and P are equivalent;Under P ,the process $\tilde{B}(t), 0\le t\le T$, is a Brownian motion,and we have

$$\frac{\mathrm{d}H(t)}{H(t)}=r(t)\mathrm{d}t+\sigma(t)\mathrm{d}\tilde{B}(t).$$

That is: $H(t)=H\exp\{\int_0^t(r(s)-\frac{1}{2}\sigma^2(s))\mathrm{d}s+\int_0^t\sigma(s)\mathrm{d}\tilde{B}(s)\}.$

For convenience, assume that $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-(1/2)\cdot s^2}\,\mathrm{d}s$ represents normal distribution function, $\mathrm{I}_A(\omega)=\begin{cases}1, & if\,\omega\in A\\ 0, & if\,\omega\notin A\end{cases}$; $N(0,1)$ represents normal distribution with mean 0 and variance 1, $\mathrm{E}(\cdot)$ represents mathematical expectation.

Let $X=\frac{\int_0^T\sigma(t)\mathrm{d}B(t)}{\sqrt{\kappa}}$, $\kappa=\int_0^T\sigma^2(t)\mathrm{d}t$, then $X\sim\mathrm{N}(0,1)$.

Let: $A=\{M(T)>\alpha H(T), M(T)-\alpha H(T)<k_1A_0\}$, $B=\{M(T)-\alpha H(T)\ge k_1A_0\}$.

According to martingale pricing method,the value of housing mortgage loan co-insurance satisfies：

$$\begin{aligned}V_0&=\mathrm{E}^{\widetilde{\mathrm{P}}}[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}(M-\alpha H(T))\mathrm{I}_A]+\mathrm{E}^{\widetilde{\mathrm{P}}}[\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}(k_1A_0+k_2(M-\alpha H(T)-k_1A_0))\mathrm{I}_B]\\&=M\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\mathrm{E}^{\widetilde{\mathrm{P}}}[\mathrm{I}_A]-\alpha\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\mathrm{E}^{\widetilde{\mathrm{P}}}[H(T)\mathrm{I}_A]\\&\quad+(k_1A_0-k_1k_2A_0+k_2\mathrm{M})\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\mathrm{E}^{\widetilde{\mathrm{P}}}[\mathrm{I}_B]-\alpha k_2\mathrm{e}^{-\int_0^T r(t)\mathrm{d}t}\mathrm{E}^{\widetilde{\mathrm{P}}}[H(T)\mathrm{I}_B]\end{aligned}\tag{5}$$

To compute V_0 ,we first compute set A.

$$\begin{aligned}A&=\{M(T)>\alpha H(T), M(T)-\alpha H(T)<k_1A_0\}\\&=\left\{\ln(\frac{M-k_1A_0}{\alpha H})-\int_0^T r(t)dt+\frac{1}{2}\kappa<\int_0^T\sigma(t)\mathrm{d}B(t)<\ln(\frac{M}{\alpha H})-\int_0^T r(t)dt+\frac{1}{2}\kappa\right\}\\&=\left\{d_1<\frac{\int_0^T\sigma(t)\mathrm{d}B(t)}{\sqrt{\kappa}}<d_2\right\}=\{d_1<X<d_2\}\end{aligned}$$

Similarly, $B=\{M(T)-\alpha H(T)\ge k_1A_0\}=\{X\le d_1\}$.

Then,we have: $\mathrm{E}^{\widetilde{\mathrm{P}}}[\mathrm{I}_A]=\Phi(d_2)-\Phi(d_1)$ (6)

$\mathrm{E}^{\widetilde{\mathrm{P}}}[\mathrm{I}_B]=\Phi(d_1)$. (7)

$$e^{-\int_0^T r(t)\mathrm{d}t}\widetilde{\mathrm{E}^{\mathrm{P}}}[H(T)\mathrm{I}_A] = He^{-\int_0^T r(t)\mathrm{d}t}\, e^{\int_0^T r(t)-\frac{1}{2}\sigma^2(t)\mathrm{d}t}\, \widetilde{\mathrm{E}^{\mathrm{P}}}[e^{\int_0^T \sigma(t)\mathrm{d}B(t)}\mathrm{I}_{\{d_1<X<d_2\}}]$$

$$= He^{-\frac{1}{2}\kappa}\widetilde{\mathrm{E}^{\mathrm{P}}}[e^{X\sqrt{\kappa}}\mathrm{I}_{\{d_3<X<d_1\}}] = H[\Phi(\sqrt{\kappa}-d_1)-\Phi(\sqrt{\kappa}-d_2)] \tag{8}$$

$$e^{-\int_0^T r(t)\mathrm{d}t}\widetilde{\mathrm{E}^{\mathrm{P}}}[H(T)\mathrm{I}_B] = He^{-\int_0^T r(t)\mathrm{d}t}\, e^{\int_0^T r(t)-\frac{1}{2}\sigma^2(t)\mathrm{d}t}\, \widetilde{\mathrm{E}^{\mathrm{P}}}[e^{\int_0^T \sigma(t)\mathrm{d}B(t)}\mathrm{I}_{\{X\le d_1\}}]$$

$$= He^{-\frac{1}{2}\kappa}\widetilde{\mathrm{E}^{\mathrm{P}}}[e^{X\sqrt{\kappa}}\mathrm{I}_{\{-X\ge -d_1\}}] = H\Phi(d_1-\sqrt{\kappa}) \tag{9}$$

Inserting (6),(7),(8),(9) into (5), we obtain the following theorem.

Theorem 3.1 Assume the financial market is complete and no arbitrage.Let $M(T)$,a constant ,be the unpaid amount . Let $r(t)$,a function of t, be the risk-free rate. Let the income of co-insurance policy satisfy equation (1), and property values $H(t)$ meet equation (2), then we obtain the martingale pricing formula of mortgage common insurance as follows:

$$\begin{aligned} V_0 = {} & e^{-\int_0^T r(t)\mathrm{d}t}[M\Phi(d_2)+(k_1A_0-k_1k_2A_0+k_2\mathrm{M}-M)\Phi(d_1)] \\ & -\alpha H[\Phi(\sqrt{\kappa}-d_1)-\Phi(\sqrt{\kappa}-d_2)]-\alpha Hk_2\Phi(d_1-\sqrt{\kappa}) \end{aligned}$$

where $d_1 = \frac{\ln(\frac{M-k_1A_0}{\alpha H})-\int_0^T r(t)dt+\frac{1}{2}\kappa}{\sqrt{\kappa}}$, $d_2 = \frac{\ln(\frac{M}{\alpha H})-\int_0^T r(t)dt+\frac{1}{2}\kappa}{\sqrt{\kappa}}$, $\kappa = \int_0^T \sigma^2(t)\mathrm{d}t$, $\Phi(x)$ represents normal distribution function.

4 Insurance Actuary Pricing of Housing Mortgage Common Insurance

Theorem 4.1 Assume that the income of co-insurance policy can be expressed as (1), $M(T)$ is the unpaid amount , $r(t)$ is the risk-free rate , property values $H(t)$ meets equation (2), then we obtain the insurance actuary pricing formula of housing mortgage common insurance as follows:

$$\begin{aligned} V_0^* = {} & e^{-\int_0^T r(t)\mathrm{d}t}[M\Phi(d_3)+(k_1A_0-k_1k_2A_0+k_2\mathrm{M}-M)\Phi(d_4)] \\ & -\alpha H\, e^{\frac{1}{2}c_1^2-c_2}[\Phi(d_3-c_1)-\Phi(d_4-c_1)]-\alpha Hk_2\, e^{\frac{1}{2}c_1^2-c_2}\,\Phi(d_4-c_1) \end{aligned} \tag{10}$$

where $d_3 \triangleq \frac{-\int_0^T r(t)\mathrm{d}t+\mathrm{In}\frac{M}{\alpha H}+\frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}\mathrm{d}t}{e^{-aT}\sqrt{\int_0^T \sigma^2(t)e^{2at}\mathrm{d}t}}$, $d_4 \triangleq \frac{-\int_0^T r(t)\mathrm{d}t+\mathrm{In}\frac{M-k_1A_0}{\alpha H}+\frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}\mathrm{d}t}{e^{-aT}\sqrt{\int_0^T \sigma^2(t)e^{2at}\mathrm{d}t}}$,

$c_1 \triangleq e^{-aT}\sqrt{\int_0^T \sigma^2(t)e^{2at}\mathrm{d}t}$, $c_2 \triangleq \frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}\mathrm{d}t$, $\Phi(\cdot)$ represents normal distribution function.

Proof: For convenience,let:

$$C \triangleq \{e^{-\int_0^T r(t)dt} M(T) > e^{-\int_0^T \beta(t)dt}\alpha H(T), e^{-\int_0^T r(t)dt} M(T) - e^{-\int_0^T \beta(t)dt}\alpha H(T) < e^{-\int_0^T r(t)dt} k_1 A_0\},$$

$$D \triangleq \{e^{-\int_0^T r(t)dt} M(T) - e^{-\int_0^T \beta(t)dt}\alpha H(T) \geq e^{-\int_0^T r(t)dt} k_1 A_0\}, \quad \xi \triangleq \frac{\int_0^T \sigma(t)e^{at} dB(t)}{\sqrt{\int_0^T \sigma^2(t)e^{2at}dt}}, \xi \sim N(0,1)$$

The expected rate of return $\int_0^T \beta(t)dt$ meets:

$$e^{\int_0^T \beta(t)dt} \triangleq \frac{EH(T)}{H} = H^{e^{-aT}-1}\exp\{\int_0^T [\mu(t) - \frac{1}{2}\sigma^2(t)]e^{-at}dt + \frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}dt\}.$$

According to the method of insurance actuary pricing, the value of housing mortgage loan co-insurance V_0^* satisfies:

$$V_0^* = E[e^{-\int_0^T r(t)dt} M I_C] - E[e^{-\int_0^T \beta(t)dt}\alpha H(T)) I_C] + E[e^{-\int_0^T r(t)dt}(k_1A_0 + k_2M - k_1k_2A_0) I_D]$$
$$-\alpha k_2 E[e^{-\int_0^T \beta(t)dt} H(T)) I_D].$$

First compute set D,

$$D = \{-\int_0^T \beta(t)\,dt + \text{In}\, H(T) \geq -\int_0^T r(t)\,dt + \text{In}\frac{M - k_1A_0}{\alpha H}\}$$
$$= \{e^{-aT}\int_0^T \sigma(t)e^{at}dB(t)\} \leq -\int_0^T r(t)\,dt + \text{In}\frac{M - k_1A_0}{\alpha H} + \frac{1}{2}\int_0^T \sigma^2(t)e^{-at}dt\}$$

$$= \{\frac{\int_0^T \sigma(t)e^{at}dB(t)}{\sqrt{\int_0^T \sigma^2(t)e^{2at}dt}} \leq \frac{-\int_0^T r(t)\,dt + \text{In}\frac{M - k_1A_0}{\alpha H} + \frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}dt}{e^{-aT}\sqrt{\int_0^T \sigma^2(t)e^{2at}dt}}\} = \{\xi \leq d_4\}$$

Similarly,

$$C = \left\{-\int_0^T r(t)\,dt + \text{In}\frac{M - k_1A_0}{\alpha H} + \frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}dt < \int_0^T \sigma(t)e^{at}dB(t) < -\int_0^T r(t)\,dt + \text{In}\frac{M}{\alpha H} + \frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}dt\right\}$$

$$= \left\{d_4 < \frac{\int_0^T \sigma(t)e^{at}dB(t)}{\sqrt{\int_0^T \sigma^2(t)e^{2at}dt}} < d_3\right\} = \{d_4 < \xi < d_3\},$$

So that, $E[I_A] = \Phi(d_3) - \Phi(d_4)$ (11)

$E[I_B] = \Phi(d_4)$ (12)

On the other hand, let $c_1 \triangleq e^{-aT}\sqrt{\int_0^T \sigma^2(t)e^{2at}dt}$, $c_2 \triangleq \frac{1}{2}\int_0^T \sigma^2(t)e^{-2at}dt$,we have

$$\mathrm{E}[\mathrm{e}^{-\int_0^T \beta(t)\mathrm{d}t} H(T)\mathrm{I}_C] = H\mathrm{E}[\mathrm{e}^{(c_1\xi - c_2)}\,\mathrm{I}_{\{d_4<\xi<d_3\}}] = H\int_{d_4}^{d_3} \frac{1}{\sqrt{2\pi}} e^{(c_1 x - c_2 - \frac{x^2}{2})}\mathrm{d}x$$

$$= H\,\mathrm{e}^{\frac{1}{2}c_1^2 - c_2}\int_{d_4-c_1}^{d_3-c_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\mathrm{d}u = H\,\mathrm{e}^{\frac{1}{2}c_1^2-c_2}[\Phi(d_3-c_1)-\Phi(d_4-c_2)] \quad (13)$$

$$\mathrm{E}[\mathrm{e}^{-\int_0^T \beta(t)\mathrm{d}t} H(T)\mathrm{I}_D] = H\mathrm{E}[\mathrm{e}^{(c_1\xi - c_2)}\,\mathrm{I}_{\{\xi\le d_4\}}] = H\int_{-\infty}^{d_4} \frac{1}{\sqrt{2\pi}} e^{(c_1 x - c_2 - \frac{x^2}{2})}\mathrm{d}x$$

$$= H\,\mathrm{e}^{\frac{1}{2}c_1^2 - c_2}\int_{-\infty}^{d_4-c_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\mathrm{d}u = H\,\mathrm{e}^{\frac{1}{2}c_1^2-c_2}\,\Phi(d_4-c_1) \quad (14)$$

In conclusion, the theorem is proved.

5 Conclusion

The traditional martingale pricing technique has often supposed that the financial market is no arbitrage and complete. On the circumstances of martingale pricing technique, a stock(or derivative securities) present price can be get from the discounted future expected cash flow, and expected value discount can be carried out under the risk neutral. If the market were arbitrage (such as the price of risky assets is driven by geometric fractional Brown motion) and incomplete (such as the price of risky assets is driven by levy process), the equivalent martingale measure is not existed or existed but not unique at this time, thus, using traditional martingale pricing technique is difficult. In 1998, Bladt and Rydberg [4] were first put forward the option pricing of insurance actuary technique. Insurance actuary technique is transfer option pricing issue to equivalent and fair premium issue. Owing to no economic hypothesizes, so it was not only effect on no-arbitrage, equivalent and complete market, but also effect on arbitrage, non-equivalent and incomplete market.

Comparing theorem 3.1 to theorem 4.1, we can see that insurance actuary pricing and traditional martingale pricing (aka no arbitrage pricing) have obvious different under the same market model. When housing price is obey to the process of O-U index, the insurance actuary pricing was related to the non-linear excursion of modulus a and the rate of fluctuation $\sigma(t)$, no arbitrage pricing was only related to the rate of fluctuation $\sigma(t)$. Due to the equivalent martingale measure is existed and unique, so the insurance have unique no arbitrage pricing. So the insurance actuary pricing is essentially a arbitrage pricing.

In short, when real estate price keep to some random processes, the guarantee insurance pricing that have offered by insurance actuary pricing technique and no arbitrage pricing that have offered by martingale pricing technique might be different.

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