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On characterizing some mixtures of probability distributions

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Abstract

The concept of recurrence relations is used to characterize mixtures of two exponential families, two 1st type Ouyang's distributions and two 2nd type Ouyang's distributions. Our results are used to deduce conclusions concerning some mixtures of some well known distributions like Burr, Pareto, Power Weibull, 1sttype Pearsonian and Ferguson's distributions. Furthermore, some characterizations related to some recently distributions like, mixtures of two exponentiated Weibull, exponentiated Pareto and generalized exponential distributions are derived from our results. In addition, some well-known results follow from our results as special cases.

Mathematics Subject Classification: 62E10

Keywords: Mixtures of distributions; truncated moments; failure rate; reversed failure rate; recurrence relations; exponentiated Burr; exponentiated Weibull; generalized exponential; Ouyang's distribuyions and Ferguson's distributions

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1 Introduction

Theory of characterizations is a very vital and interesting branch of science which plays an important role in many fields such as mathematical statistics, reliability, statistical inference, theory of mixtures and actuarial science. It concerns with the characteristic properties of the probability distributions so that it helps researchers to distinguish them and in most cases determine distributions uniquely. Some excellent references are Azlarov and Volodin [21], Kagan, Linnik and Rao [2] and Galambos and Kotz [15], among others.

Several ideas and concepts have been used to identify different types of the probability distributions. In many practical situations, researchers may have reasons to assume the knowledge of the expected value of the random variable under study. This has motivated several authors and scientists to characterize distributions using the concept of conditional expectations. Gupta [19], Ouyang [16], Talwalker [20] and Elbatal et al.[13] have used the concept of right truncated moments to characterize various types of distributions like Weibull, gamma, beta, exponential, Burr, Pareto and Power distributions.

Actually characterizations via right truncated moments are very important in practice since, in some situations some measuring devices may be unable to record values greater than time t. On the other hand, there are some measuring devices which may not be able (or fail) to record values smaller than time t. This has encouraged several authors to deal with the problem of characterizing distributions by means of left truncated moments, see, e.g., Glänzel [22],Gupta [19], Ahmed [4], Dimaki and Xekalaki [7] and Fakhry [18].

The idea of recurrence relations is another important concept that has been used many times to characterize different types of distributions. In fact, a recurrence relation is a relation in which the function of interest , f_n , is defined in terms of a smaller value of n. Recurrence relations may be used to minimize the number of steps required to obtain a general form for the function under consideration. Furthermore, recurrence relations together with some initial conditions define functions uniquely. This has motivated several authors to use this concept in characterizing distributions see, e.g., Khan et. al. [3], Lin [11], Fakhry [18], Ahmad [1], and Al-Hussaini et. al. [10] .

In this paper, we are interested in characterizing mixtures of some classes of probability distributions using the concepts of differential equations, recurrence relations and truncated moments of some function $h(X)$ of the random variable X.

2 Main Results

Mixtures of probability distributidons play a vital role in statistics, reliability, life testing, fisheries research, economics, medicine and psychology, among others. Some excellent references are Everitt and Hand [5], Titterington et al.[9], Lindsay [6], Maclachlan and Peel [12] and B$\ddot{o}$hning [8].

The following Theorem characterizes a mixture of two exponential families using the concept of right truncated moment.

Theorem 2.1 Let X be a continuous random variable with density function $f(\cdot)$, cdf $F(\cdot)$, failure rate $\omega(\cdot)$ and reversed failure rate $\mu(\cdot)$ such that $F(\cdot)$ has non vanishing first order derivative on (α,β) so that in particular $0 \le F(x) \le 1$ for all x.Assume that $h(\cdot)$ is a differentiable function defined on (α,β) such that:

(a) $E(h^k(X)) < \infty$ for every natural number k .

(b) $h'(x) \neq 0, \quad \forall x \in (\alpha,\beta)$.

(c) $\lim_{x\to\alpha^+} h(x) = h(\alpha)$

(d) $\lim_{x\to\beta^-} h(x) = \infty$

Then the following statements are equivalent:

(2.1) $$F(x) = \sum_{i=1}^{2} \rho_i \left[1 - \exp-\left(\frac{h(x)-h(\alpha)}{\theta_i}\right)\right], \quad \forall x \in (\alpha,\beta).$$

Where ρ_1 and ρ_2 are positive real numbers such that $\sum_{i=1}^{2}\rho_i = 1$

(2.2) $$\frac{\theta_1\theta_2}{h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' + \left(\theta_1+\theta_2\right)\frac{F'(x)}{h'(x)} + F(x) = 1,\ \forall\ x\in(\alpha,\beta).$$

(2.3) $$\begin{aligned} m_k &= E(h^k(X)\,|\,X<y) \\ &= -\frac{\mu(y)}{\omega(y)}h^k(y) + k\theta_1\theta_2\frac{\mu(y)}{h'(y)}[h(y)]^{k-1} + k\left(\theta_1+\theta_2\right)m_{k-1} \\ &\quad - k(k-1)\theta_1\theta_2 m_{k-2} + \frac{[h(\alpha)]^{k-1}}{F(y)}\left[h(\alpha) - k\theta_1\theta_2\frac{F'(\alpha)}{h'(\alpha)}\right] \end{aligned}$$

Proof. $1 \Rightarrow 2$

We have from (2.1) the following results:

(a) $$\frac{F'(x)}{h'(x)} = \sum_{i=1}^{2}\frac{\rho_i}{\theta_i}\exp-[\frac{h(x)-h(\alpha)}{\theta_i}]$$

(b) $$\left(\frac{F'(x)}{h'(x)}\right)' = -h'(x)\sum_{i=1}^{2}\frac{\rho_i}{\theta_i^2}\exp-[\frac{h(x)-h(\alpha)}{\theta_i}]$$

Therefore,

$$\begin{aligned} &\frac{\theta_1\theta_2}{h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' + \left(\theta_1+\theta_2\right)\frac{F'(x)}{h'(x)} + F(x) \\ &\quad = -\theta_1\theta_2\sum_{i=1}^{2}\frac{\rho_i}{\theta_i^2}\exp-[\frac{h(x)-h(\alpha)}{\theta_i}] + \left(\theta_1+\theta_2\right)\sum_{i=1}^{2}\frac{\rho_i}{\theta_i}\exp-[\frac{h(x)-h(\alpha)}{\theta_i}] \\ &\quad\quad + \sum_{i=1}^{2}\rho_i[1-\exp-[\frac{h(x)-h(\alpha)}{\theta_i}] = 1 \end{aligned}$$

$2 \Rightarrow 3$

We have, by definition:

(2.4) $$m_k = E(h^k(X)|X<y) = \frac{\int_\alpha^y h^k(x)dF(x)}{F(y)} = h^k(y) - \frac{k\int_\alpha^y h^{k-1}(x)h'(x)F(x)dx}{F(y)}$$

Let $J = k\int_{\alpha}^{y} h^{k-1}(x)h'(x)F(x)dx$.

Eliminating F(x), using equation (2.2), we have:

$$J = k\int_{\alpha}^{y} h^{k-1}(x)h'(x)dx - k\left(\theta_1+\theta_2\right)\int_{\alpha}^{y} h^{k-1}(x)F'(x)dx - k\theta_1\theta_2\int_{\alpha}^{y} h^{k-1}(x)\left(\frac{F'(x)}{h'(x)}\right)' dx$$

Integrating by parts and making use of the definition of m_k, one gets:

$$\begin{aligned} J &= h^k(y) - h^k(\alpha) - k(\theta_1+\theta_2)m_{k-1}F(y) - k\theta_1\theta_2\int_{\alpha}^{y} h^{k-1}(x)\left(\frac{F'(x)}{h'(x)}\right)' dx \\ &= h^k(y) - h^k(\alpha) - k(\theta_1+\theta_2)m_{k-1}F(y) - k\theta_1\theta_2 h^{k-1}(y)\frac{F'(y)}{h'(y)} \\ &\quad + k\theta_1\theta_2 h^{k-1}(\alpha)\frac{F'(\alpha)}{h'(\alpha)} + k(k-1)\theta_1\theta_2\int_{\alpha}^{y} h^{k-2}(x)F'(x)dx \\ &= h^k(y) - k(\theta_1+\theta_2)m_{k-1}F(y) - k\theta_1\theta_2 h^{k-1}(y)\frac{F'(y)}{h'(y)} \\ &\quad + k(k-1)\theta_1\theta_2 m_{k-2}F(y) - h^{k-1}(\alpha)\left[h(\alpha) - k\theta_1\theta_2\frac{F'(\alpha)}{h'(\alpha)}\right]. \end{aligned}$$

Substituting these results in equation (2.4), recalling that:

(a) $\omega(y) = \dfrac{F'(y)}{1-F(y)}$ (b) $\mu(y) = \dfrac{F'(y)}{F(y)}$

One gets:

$$\begin{aligned} m_k &= -\frac{\mu(y)}{\omega(y)}h^k(y) + k\theta_1\theta_2\frac{\mu(y)}{h'(y)}h^{k-1}(y) + k\left(\theta_1+\theta_2\right)m_{k-1} \\ &\quad - k\left(k-1\right)\theta_1\theta_2 m_{k-2} + \frac{h^{k-1}(\alpha)}{F(y)}[h(\alpha) - k\theta_1\theta_2\frac{F'(\alpha)}{h'(\alpha)}]. \end{aligned}$$

$3 \Rightarrow 1$

Equation (2.3) can be written in integral form as follows:

$$\begin{aligned} \int_{\alpha}^{y} h^k(x)dF(x) &= -[1-F(y)]h^k(y) + k\theta_1\theta_2 h^{k-1}(y)\frac{F'(y)}{h'(y)} + k(\theta_1+\theta_2)\int_{\alpha}^{y} h^{k-1}(x)dF(x) \\ &\quad - k(k-1)\theta_1\theta_2\int_{\alpha}^{y} h^{k-2}dF(x) - h^{k-1}(\alpha)\left[k\theta_1\theta_2\frac{F'(\alpha)}{h'(\alpha)} - h(\alpha)\right] \end{aligned}$$

Differentiating both sides of the last equation with respect to y, adding to both sides the term "$-h^k(y)F'(y)$", cancelling out the term "$k(k-1)\theta_1\theta_2 h^{k-2}(y)F'(y)$" from the right side and dividing the result by "$kh'(y)h^{k-1}(y)$", one gets:

(2.5) $$\frac{\theta_1\theta_2}{h'(y)}\left(\frac{F'(y)}{h'(y)}\right)' + (\theta_1+\theta_2)\frac{F'(y)}{h'(y)} + F(y) = 1.$$

This is a non linear differential equation of the second order with variable coefficient. To solve it put $z = h(y) - h(\alpha)$, then

$$F'(y) = \frac{dF(y)}{dy} = \frac{dF(z)}{dz}\frac{dz}{dy} = h'(y)F'(z)$$

$$\left(\frac{F'(y)}{h'(y)}\right)' = \frac{d}{dy}\frac{F'(y)}{h'(y)} = \frac{dF'(z)}{dy} = \frac{dF'(z)}{dz}\frac{dz}{dy} = h'(y)F''(z).$$

Substituting these results in equation (2.5), we get:

(2.6) $$\theta_1\theta_2 F''(z) + (\theta_1+\theta_2)F'(z) + F(z) = 1$$

Set $F(z) = 1 - G(z)$, equation (2.6) becomes:

$$\theta_1\theta_2 G''(z) + (\theta_1+\theta_2)G'(z) + G(z) = 0.$$

The solution of this differential equation is known to be (see, Ross [17]):

$$G(z) = \sum_{i=1}^{2}\rho_i \exp -\frac{z}{\theta_i}$$

Therefore

$$F(y) = \sum_{i=1}^{2}\rho_i\left[1 - \exp -\left(\frac{h(y)-h(\alpha)}{\theta_i}\right)\right].$$

The fact that $\lim_{x\to\beta^-} F(x) = 1$ and the conditions imposed on the function $h(x)$ imply that: $\rho_1 + \rho_2 = 1$, while the fact that $0 \le F(x) \le 1$ implies that $0 \le \rho_i \le 1$, for $i \in \{1,2\}$.

Remarks 2.1

(1) If we set $h(X) = X^b$, $b > 0$, $\alpha = 0$, $\beta = \infty$, we obtain characterizations concerning a mixture of two Weibull distributions with respective parameters b, θ_1 and b , θ_2. For $b = 1$, the results reduce to that of a mixture of two exponential distributions. For $b = 1$, $k = 1$ and $\theta_1 = \theta_2 = \theta$, result (2.3) reduces to that of Talwalker [20], concerning the exponential distribution, namely, $E(X|X < y) = \frac{-(1-F(y)}{F(y)} y + \theta$, iff X follows the exponential distribution with parameter θ.

(2) If we set $h(X) = \ln{(1 + X^b)}$, $b > 0, \theta_i = \frac{1}{k_i}$, $k_i \notin \{0,1\}$, $i \in \{1,2\}, \alpha = 0$ and $\beta = \infty$, we obtain characterizations concerning a mixture of two Burr distributions with respective parameters b, k_1 and b, k_2. For $b = 1$, we have characterizations concerning a mixture of two Pareto distributions. For $k_1 = k_2 = k$, we have characterizations concerning Burr distribution.

(3) If we set $h(X) = -ln\left(\frac{\beta-X}{\beta-\alpha}\right), \theta_i = \frac{1}{k_i}$, $i \in \{1,2\}$, we obtain characterizations concerning a mixture of two 1st type Pearsonian distributions with respective parameters β, α, k_1 and β, α, k_2.For $\beta = 1, \alpha = 0$, we have results concerning a mixture of two beta distributions with respective parameters $1, k_1$and$1, k_2$.

Now, we use the concept of right truncated moments to identify a mixture of two 2nd type Ouyang's distributions [16] with parameters d, c_1 and d, c_2.

Theorem 2.2 Let X be a continuous random variable with density function $f(\cdot)$,cdf $F(\cdot)$, failure rate $\omega(\cdot)$ and reversed failure rate function $\mu(\cdot)$ such that $F'(x) > 0$ for all $x \in (\alpha, \beta)$ so that in particular $0 \leq F(x) \leq 1$ for all x. Assume that $h(\cdot)$ is a differentiable function defined on (α, β) such that:

(a) $E(h^k(X)) < \infty$ for every natural number k.

(b) $h'(x) \neq 0 \quad \forall x \in (\alpha, \beta)$

(c) $\lim_{x\to\alpha^+} h(x) = 1-d$.

(d) $\lim_{x\to\beta^-} h(x) = -d$.

Then the following statements are equivalent:

(2.7) $F(x)=\sum_{i=1}^{2}\rho_i\left[1-\left(h(x)+d\right)^{c_i}\right], \quad \forall x \in (\alpha,\beta),$

where d and c_i are parameters such that $c_i \notin \{0,-1\}$.

(2.8) $$\frac{[h(x)+d]^2}{c_1c_2h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' - \frac{(c_1+c_2-1)}{c_1c_2}\left[h(x)+d\right]\frac{F'(x)}{h'(x)}+F(x)=1$$

(2.9) $$\begin{aligned} m_k &= E(h^k(X) \mid X<y) \\ &= \delta\Bigg[-c_1c_2\frac{\mu(y)}{\omega(y)}h^k(y)+k\left[h(y)+d\right]^2\frac{\mu(y)}{h'(y)}h^{k-1}(y) \\ &\quad -kd\left(c_1+c_2+2k-1\right)m_{k-1}-k\left(k-1\right)d^2m_{k-2} \\ &\quad +\frac{h^{k-1}(\alpha)}{F(y)}[c_1c_2h(\alpha)-k\frac{F'(\alpha)}{h'(\alpha)}]\Bigg] \end{aligned}$$

where $\delta^{-1}=c_1c_2+k(c_1+c_2+k)$.

Proof.

$1 \Rightarrow 2$

From equation (2.7), we have the following results:

(a) $F'(x)=-h'(x)\sum_{i=1}^{2}\rho_i c_i[h(x)+d]^{C_i-1}$, and

(b) $\left(\frac{F'(x)}{h'(x)}\right)' = -h'(x)\sum\rho_i c_i\left(c_i-1\right)[h(x)+d]^{c_i-2}$, therefore

$$\frac{[h(x)+d]^2}{c_1c_2h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' - \frac{(c_1+c_2-1)}{c_1c_2}\left[h(x)+d\right]\frac{F'(x)}{h'(x)}+F(x)$$
$$=\sum_{i=1}^{2}\frac{\rho_i c_i}{c_1c_2}(c_1+c_2-c_i)(h(x)+d)^{c_i}+\sum_{i=1}^{2}\rho_i[1-(h(x)+d)^{c_i}]=1$$

$2 \Rightarrow 3$

By definition we have

(2.10) $$m_k = \int_\alpha^y \frac{h^k(x)dF(x)}{F(y)} = h^k(y) - k\int_\alpha^y \frac{[h(x)]^{k-1}h'(x)F(x)dx}{F(y)}$$

Making use of equation (2.8) to eliminate F(x), the second part of equation (2.10) becomes:

$$J = k\int_\alpha^y \frac{[h(x)]^{k-1}h'(x)}{F(y)}dx + \frac{k(c_1+c_2-1)}{c_1c_2F(y)}\int_\alpha^y [h(x)+d][h(x)]^{k-1}F'(x)dx$$
$$-\frac{k}{c_1c_2F(y)}\int_\alpha^y [h(x)+d]^2[h(x)]^{k-1}\left(\frac{F'(x)}{h'(x)}\right)' dx$$

Integrating by parts and using the definition of m_k as well as the assumptions imposed on the function $h(\cdot)$, one gets:

(2.11) $$J = \frac{h^k(y)-h^k(\alpha)}{F(y)} + \frac{(c_1+c_2-1)k}{(c_1c_2)}[m_k + dm_{k-1}]$$
$$-\frac{k}{c_1c_2F(y)}\int_\alpha^y [h(x)+d]^2[h(x)]^{k-1}\left(\frac{F'(x)}{h'(x)}\right)' dx$$

Let $$Q = \frac{k}{c_1c_2F(y)}\int_\alpha^y \left[h(x)+d\right]^2[h(x)]^{k-1}\left(\frac{F'(x)}{h'(x)}\right)' dx.$$

Integrating by parts, taking into consideration, the definition of m_k and the conditions imposed on the function $h(\cdot)$, one gets:

(2.12) $$Q = \frac{k}{c_1c_2}[h(y)+d]^2[h(y)]^{k-1}\frac{\mu(y)}{h'(y)} - \frac{k}{c_1c_2F(y)}[h(\alpha)]^{k-1}\frac{F'(\alpha)}{h'(\alpha)}$$
$$-\frac{2k}{c_1c_2}[m_k + dm_{k-1}] - \frac{k(k-1)}{c_1c_2}[m_k + 2dm_{k-1} + d^2m_{k-2}]$$

Substituting these results in equation (2.10) and collecting similar terms, we get:

$$m_k = -\frac{1-F(y)}{F(y)}h^k(y) + \frac{k}{c_1c_2}[h(y)+d]^2\frac{\mu(y)}{h'(y)}[h(y)]^{k-1} - \frac{k(c_1+c_2+k)}{c_1c_2}m_k$$
$$-\frac{kd}{c_1c_2}(c_1+c_2+2k-1)m_{k-1} - \frac{k(k-1)d^2}{c_1c_2}m_{k-2} + \frac{[h(\alpha)]^{k-1}}{F(y)c_1c_2}[c_1c_2h(\alpha) - k\frac{F'(\alpha)}{h'(\alpha)}].$$

Solving the last equation for m_k , we get

$$m_k = \delta\left\{ -c_1c_2\frac{\mu(y)}{\omega(y)}h^k(y) + k\left[h(y)+d\right]^2\frac{\mu(y)}{h'(y)}h^{k-1}(y) - kd\left(c_1+c_2+2k-1\right)m_{k-1} - k\left(k-1\right)d^2 m_{k-2} + \frac{[h(\alpha)]^{k-1}}{F(y)}[c_1c_2h(\alpha) - k\frac{F'(\alpha)}{h'(\alpha)}]\right\},$$

where $\delta^{-1} = c_1c_2 + k(c_1+c_2+k).$

$3 \Rightarrow 1$

Equation (2.9) can be written in integral form as follows:

$$[c_1c_2 + k(c_1+c_2+k)]\int_\alpha^y h^k(x)dF(x) = -c_1c_2(1-F(y))h^k(y) + k[h(y)+d]^2[h(y)]^{(k-1)}\frac{F'(y)}{h'(y)} - kd[c_1+c_2+2k-1]\int_\alpha^y [h(x)]^{k-1}dF(x) - k(k-1)d^2\int_\alpha^y [h(x)]^{k-2}dF(x) + c_1c_2h^k(\alpha) - k[h(\alpha)]^{k-1}\frac{F'(\alpha)}{h'(\alpha)}. \tag{2.13}$$

Differentiating equation (2.13) with respect to *y*, cancelling out the terms "$c_1c_2h^k(y)F'(y)$" from both sides, removing the term "$k(k-1)d^2[h(y)]^{k-2}F'(y)$" from the right side and collecting similar terms together then dividing the result by "$kc_1c_2h'(y)\left[h(y)\right]^{k-1}$", one gets:

$$\frac{[h(y)+d]^2}{c_1c_2h'(y)}\left(\frac{F'(y)}{h'(y)}\right)' - \frac{\left(c_1+c_2-1\right)}{c_1c_2}\left[h(y)+d\right]\frac{F'(y)}{h'(y)} + F(y) = 1 \tag{2.14}$$

To solve this equation, set $z = h(y) + d$, then

$$F'\left(y\right) = \frac{dF(y)}{dy} = \frac{dF(z)}{dz}\frac{dz}{dy} = h'\left(y\right)F'(z)$$

and
$$\left(\frac{F'(y)}{h'(y)}\right)' = \frac{d}{dy}\frac{F'(y)}{h'(y)} = \frac{dF'(z)}{dy} = \frac{dF'(z)}{dz}\frac{dz}{dy} = h'(y)F''(z).$$

Substituting these results into equation (2.14), we get:

(2.15) $$\frac{z^2}{c_1c_2}F''(z)-\frac{c_1+c_2-1}{c_1c_2}zF'(z)+F(z)=1.$$

Set $z=e^x$, then

$$F'(z)=\frac{dF(z)}{dz}=\frac{dF(x)}{dx}\frac{dx}{dz}=\frac{F'(x)}{z}$$

$$F''(z)=\frac{dF'(z)}{dz}=\frac{d}{dz}\frac{F'(x)}{z}=\frac{1}{z}\frac{d}{dz}F'(x)-\frac{F'(x)}{z^2}=\frac{F''(x)-F'(x)}{z^2}.$$

Substituting these results in equation (2.15), we get:

$$\frac{F''(x)}{c_1c_2}-\frac{c_1+c_2}{c_1c_2}F'(x)+F(x)=1.$$

Putting $F(x)=1-G(x)$, the last equation becomes:

$$\frac{G''(x)}{c_1c_2}-\left[\frac{1}{c_1}+\frac{1}{c_2}\right]G'(x)+G(x)=0$$

Therefore $G(x)=\rho_1\exp(c_1x)+\rho_2\exp(c_2x)$.

Hence, $F(y)=\sum_{i=1}^{2}\rho_i\left[1-[h(y)+d]^{c_i}\right].$

The fact that $\lim_{y\to\beta^-}F(y)=1$ as well as the conditions imposed on the function $h(y)$ give $\rho_1+\rho_2=1$, while the fact that $0\le F(x)\le 1$ implies that $0\le\rho_i\le 1$ for $i\in\{1,2\}$.

This completes the proof.

Remarks 2.2

(1) If we set $h(X)=X^a$, $d=1, c_i=-b_i$, where a and b_i are positive parameters such that $b_i\notin\{0,1\}$, $i\in\{1,2\}$, $\alpha=0$ and $\beta=\infty$, we obtain characterizations concerning a mixture of two Burr distributions with respective parameters a, b_1 and a, b_2 . For $a=1$, we obtain characterizations concerning a mixture of two Pareto distributions. For $b_1=b_2=b$, we have characterizations concerning Burr distribution with parameters a, b.

(2) If we set $h(X) = \frac{-X}{\beta-\alpha}$, $d = \frac{\beta}{\beta-\alpha}$, $c_i = \theta_i > 0$, $i \in \{1,2\}$, we have characterizations concerning a mixture of two first type Pearsonian distributions with respective parameters β, α, θ_1 and β, α, θ_2 . For $\beta = 1$ and $\alpha = 0$, we have characterizations concerning a mixture of two beta distributions with respective parameters 1,θ_1 and 1, θ_2.

(3) If we set $h(X) = exp - X^b, d = 0, c_i = \frac{1}{\theta_i}$, $\alpha = 0$ and $\beta = \infty$ where b and θ_i are positive parameters , we obtain characterizations concerning a mixture of two Weibull distributions with respective positive parameters b , θ_1 and b , θ_2. For $= 2$, we have a mixture of two Rayleigh distributions with respective parameters θ_1 and θ_2 , For $\theta_1 = \theta_2 = \theta, k = 1$, result (2.9) reduces to Ouyang's result [16] concerning Weibull distribution, namely, $E\left(e^{-X^b} \middle| X < y\right) = \frac{1}{\theta+1}(e^{\frac{-y^b}{\theta}} + 1)$, iff X follows the Weibull distribution with positive parameters b and θ. For $b = 1$, we have characterizations concerning a mixture of two exponential distributions with respective parameters θ_1 and θ_2. If in addition, we have $\theta_1 = \theta_2 = \theta$, result (2.9), reduces to Talwalker's result [20].

(4) If we set $h(X) = \frac{Z(X)}{Z(\alpha)+g(n)/[m(n)-1]}$, $d = \frac{g(n)/[m(n)-1]}{Z(\alpha)+g(n)/[m(n)-1]}$ and $c_1 = c_2 = \frac{m(n)}{1-m(n)}$, where $m(\cdot)$ and $g(\cdot)$ are finite real valued functions of n and $Z(x)$ is a differentiable function defined on (α, β) such that $\lim_{x\to\alpha^+} Z(x) = Z(\alpha)$ and $\lim_{x\to\beta^-} Z(x) = \frac{-g(n)}{m(n)-1}$, we obtain characterizations concerning a class of distribution defined by Talwalker [20].

The following Theorem characterizes a mixture of two 1st type Ouyang's distributions with respective parameters d, c_1 and d, c_2.

Theorem 2.3 Let X be a continuous random variable with density function $f(\cdot)$, cdf $F(\cdot)$, failure rate function $\omega(\cdot)$ and reversed failure rate function $\mu(\cdot)$ such that $F(x)$ is a twice differentiable function on (α,β) with $F'(x) \succ 0$ for all x so that in particular $0 \le F(x) \le 1$. Assume that $h(\cdot)$ is a differentiable function defined on (α,β) such that:

(a) $E(h^k(X)) < \infty$ for every natural number k.

(b) $h'(x) \neq 0$, $\forall x \in (\alpha,\beta)$.

(c) $\lim_{x\to\alpha^+} h(x) = d$.

(d) $\lim_{x\to\beta^-} h(x) = d-1$.

Then the following statements are equivalent:

(2.19) $$F(x) = \sum_{i=1}^{2} \rho_i [d - h(x)]^{c_i}, \quad \forall x \in (\alpha,\beta),$$

where d and c_i are parameters such that $c_i \notin \{0,-1\}$ for $i \in \{1,2\}$.

(2.20) $$\frac{[d-h(x)]^2}{c_1 c_2 h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' + \frac{(c_1+c_2-1)}{c_1 c_2}\left[d-h(x)\right]\frac{F'(x)}{h'(x)} + F(x) = 0$$

(2.21) $$m_k = E\left(h^k(X) | X > y\right)$$
$$= \delta\Bigg\{ -c_1 c_2 \frac{\omega(y)}{\mu(y)} h^k(y) - \frac{k\left[d-h(y)\right]^2 \omega(y)}{h'(y)}[h(y)]^{k-1} - k(k-1)d^2 m_{k-2}$$
$$+ kd(c_1+c_2+2k-1)m_{k-1} + \frac{[h(\beta)]^{k-1}}{1-F(y)}[c_1 c_2 h(\beta) + k\frac{F'(\beta)}{h'(\beta)}]\Bigg\},$$

where $\delta^{-1} = c_1 c_2 + k(c_1 + c_2 + k)$.

Proof.

$1 \Rightarrow 2$

It easy to see that (using (2.19)):

$$F'(x) = -h'(x)\sum_{i=1}^{2} \rho_i c_i [d - h(x)]^{c_i - 1}$$

and $\left(\dfrac{F'(x)}{h'(x)}\right)' = h'(x)\sum_{i=1}^{2}\rho_i c_i (c_i - 1)[d - h(x)]^{c_i - 2}$.

Therefore

$$\frac{[d-h(x)]^2}{c_1c_2h'(x)}\left(\frac{F'(x)}{h'(x)}\right)' + \frac{(c_1+c_2-1)}{c_1c_2}[d-h(x)]\frac{F'(x)}{h'(x)} + F$$
$$= \sum_{i=1}^{2}\rho_i c_i [d-h(x)]^{c_i}\left\{\frac{c_i - c_1 - c_2}{c_1c_2}\right\} + \sum_{i=1}^{2}\rho_i [d-h(x)]^{c_i} = 0$$

$2 \Rightarrow 3$

By definition we have: $m_k = \int_y^\beta \dfrac{h^k(x)dF(x)}{1-F(y)}$

Integrating by parts, recalling that $\lim_{x\to\beta^-} F(x) = 1$, $\left[1-F(y)\right]\omega(y) = F'(y)$ and $F(y)\mu(y) = F'(y)$, one gets:

$$\text{(2.22)} \quad m_k = \frac{h^k(\beta)}{1-F(y)} - \frac{\omega(y)}{\mu(y)}h^k(y) - k\int_y^\beta \frac{[h(x)]^{k-1}h'(x)F(x)dx}{1-F(y)}.$$

Let $J = \int_y^\beta [h(x)]^{k-1}h'(x)F(x)dx.$

Eliminating $F(x)$, using equation (2.20), one gets:

$$J = -\int_y^\beta \frac{[d-h(x)]^2[h(x)]^{k-1}}{c_1c_2}\left(\frac{F'(x)}{h'(x)}\right)' dx - \frac{c_1+c_2-1}{c_1c_2}\int_y^\beta [d-h(x)][h(x)]^{k-1}F'(x)dx$$

Integrating by parts, recalling that $\lim_{x\to\beta^-}[d-h(x)]^2 = 1$ and $\lim_{x\to\beta^-} F(x) = 1$, one gets:

$$J = \frac{[h(\beta)]^{k-1}F'(\beta)}{c_1c_2h'(\beta)} + [d-h(y)]^2[h(y)]^{k-1}\frac{F'(y)}{c_1c_2h'(y)}$$
$$+\frac{k-1}{c_1c_2}\int_y^\beta [d-h(x)]^2[h(x)]^{k-2}F'(x)dx - \frac{c_1+c_2+1}{c_1c_2}\int_y^\beta [d-h(x)][h(x)]^{k-1}F'(x)dx$$

Substituting this result in equation (2.22), making use of the definition of m_k and performing some elementary computation, one gets:

$$m_k = \frac{h^k(\beta)}{1-F(y)} - \frac{\omega(y)}{\mu(y)} h^k(y) - \frac{k}{c_1c_1[1-F(y)]}[h(y)]^{k-1}[d-h(y)]^2 \frac{F'(y)}{h'(y)}$$
$$-\frac{k(k-1)}{c_1c_2}[d^2 m_{k-2} - 2dm_{k-1} + m_k] + \frac{k(c_1+c_2+1)}{c_1c_2}[dm_{k-1} - m_k]$$
$$+\frac{k[h(\beta)]^{k-1}}{c_1c_2[1-F(y)]} \frac{F'(\beta)}{h'(\beta)}$$

Solving the last equation for m_k, we get:

$$m_k = \delta\Big\{-c_1c_2 \frac{\omega(y)}{\mu(y)} h^k(y) - \frac{k[d-h(y)]^2 \omega(y)}{h'(y)}[h(y)]^{k-1} + kd(c_1+c_2+2k-1)m_{k-1}$$
$$-k(k-1)d^2 m_{k-2} + \frac{[h(\beta)]^{k-1}}{1-F(y)}[c_1c_2 h(\beta) + k\frac{F'(\beta)}{h'(\beta)}]\Big\},$$

where $\delta^{-1} = c_1c_2 + k(c_1+c_2+k)$.

$3 \Rightarrow 1$

Multiplying both sides of equation (2.21) by δ^{-1} then writing it in integral form as follows:

$$[c_1c_2 + k(c_1+c_2+k)]\int_y^\beta h^k(x)dF(x)$$
$$= -c_1c_2 F(y)h^k(y) - k[h(y)]^{k-1}[d-h(y)]^2 \frac{F'(y)}{h'(y)} + kd(c_1+c_2+2k-1)\int_y^\beta [h(x)]^{k-1} dF(x)$$
$$-k(k-1)d^2 \int_y^\beta [h(x)]^{k-2} dF(x) + [h(\beta)]^{k-1}[c_1c_2 h(\beta) + k\frac{F'(\beta)}{h'(\beta)}].$$

Differentiating both sides of the last equation with respect to y, cancelling out the term "$-c_1c_2h^k(y)F'(y)$" from both sides and dividing the result by "$-k[h(y)]^{k-2}$", one gets:

$$\left(c_1+c_2+k\right)h^2\left(y\right)F'\left(y\right)$$
$$= c_1c_2 h\left(y\right)h'\left(y\right)F\left(y\right) + \left(k-1\right)\left[d-h\left(y\right)\right]^2 F'\left(y\right)$$
$$-2h\left(y\right)\left[d-h\left(y\right)\right]F'\left(y\right) + h\left(y\right)\left[d-[h\left(y\right)\right]^2\left(\frac{F'\left(y\right)}{h'\left(y\right)}\right)'$$
$$+d\left(c_1+c_2+2k-1\right)h\left(y\right)F'\left(y\right) - \left(k-1\right)d^2F'(y)$$

Collecting similar terms, removing the term "$(k-1)d^2F'(y)$" from the right side and dividing the result by "$c_1c_2h'(y)h(y)$", one gets:

(2.23) $$\frac{[d-h(y)]^2}{c_1c_2h'(y)}\left(\frac{F'(y)}{h'(y)}\right)'+\frac{(c_1+c_2-1)}{c_1c_2}[d-h(y)]\frac{F'(y)}{h'(y)}+F(y)=0$$

This is a homogenous second order differential equation with variable coefficient. To solve it set $z=\ln[d-h(y)]$. Then

$$F'(y)=\frac{dF(y)}{dy}=\frac{dz}{dy}\frac{dF(z)}{dz}=\frac{-h'(y)}{d-h(y)}F'(z),$$

$$\left(\frac{F'(y)}{h'(y)}\right)'=\frac{d}{dy}\frac{F'(y)}{h'(y)}=\frac{d}{dy}\left(\frac{-F'(z)}{d-h(y)}\right)=-\frac{\left[d-h(y)\right]\frac{dF'(z)}{dy}+h'(y)F'(z)}{\left[d-h(y)\right]^2}$$

$$=-\frac{\left[d-h(y)\right]\frac{dz}{dy}\frac{dF'(z)}{dz}+h'(y)F'(z)}{[d-h(y)]^2}=\frac{h'(y)}{[d-h(y)]^2}[F''(z)-F'(z)].$$

Substituting these results in equation (2.23), one gets:

$$\frac{F''(z)}{c_1c_2}-\left(\frac{1}{c_1}+\frac{1}{c_2}\right)F'(z)+F(z)=0$$

Therefore

$$F(z)=\rho_1 exp(c_1z)+\rho_2 exp(c_2z)$$

Hence, $F(y)=\rho_1[d-h(y)]^{c_1}+\rho_2[d-h(y)]^{c_2}$.

The fact that $\lim_{y\to\beta^-}F(y)=1$ and the assumption that $\lim_{y\to\beta^-}h(y)=d-1$ show that $\rho_1+\rho_2=1$, while the fact that $0\le F(x)\le 1$ implies that $0\le\rho_i\le 1$, $\in\{1,2\}$. This completes the proof.

Remarks 2.3

(1) If we set $h(X)=\frac{-X}{r-b}$, $d=\frac{-b}{r-b}$, $\alpha=b$, $\beta=r$, $c_i=\theta_i>0$, $i\in\{1,2\}$, we have characterizations concerning a mixture of two Ferguson' distributions of the first type with respective parameters b, r, θ_1 and b, r, θ_2. For $b = 0$ and $r = 1$ we have characterizations concerning a mixture of two Power distributions with parameters θ_1 and θ_2. For $b = 0, r = 1$ and $\theta_1 = \theta_2 = \theta$, we obtain characterizations concerning a Power distribution with parameter θ, if , in addition, we set $\theta = 1$, we obtain results concerning the uniform distribution.

(2) If we set $h(X) = -\left(\frac{\mathrm{r-b}}{\mathrm{r-X}}\right)$, $d = 0$, $\alpha = -\infty$, $\beta = b$, $c_i = \theta_i > 0, i \in \{1,2\}$, we obtain characterizations concerning a mixture of two Ferguson's distributions of the second type with respective parameters r, b, θ_1 and r, b, θ_2.

(3) If we set $h(X) = -\exp(X - b)$, $d = 0$, $\alpha = -\infty$, $\beta = b$ and $c_i = \frac{1}{\theta_i} > 0$, $i \in \{1,2\}$, we obtain characterizations concerning a mixture of two Ferguson's distributions of the third type.

(4) If we set $h(X) = exp - \left(\frac{X}{b}\right)^a$, $d = 1$, $c_i = \theta_i$, $i \in \{1,2\}$, where a and θ_i are positive parameters, α=0 and β=∞ , we obtain characterizations concerning a mixture of two exponentiated Weibull distributions with respective positive parameters a, b, θ_1 and a, b, θ_2 . For $\theta_1 = \theta_2 = \theta$, we have results concerning the exponentiated Weibull distribution. For $\theta = 1$, we obtain characterizations concerning the Weibull distribution with parameters b, a. If in addition $a = 1$, we have characterizations concerning the exponential distribution with parameter b.

(5) If we set $h(X) = \left(\frac{b}{X}\right)^a$, $d = 1, c_i = \theta_\mathrm{i}, i \in \{1,2\}, \alpha = b$ and $\beta = \infty$, we obtain characterizations concerning a mixture of two exponentiated Pareto distributions of the first type with respective parameters b, a, θ_1 and b, a, θ_2 . For $\theta_1 = \theta_2 = \theta$, we have characterizations concerning exponentiated Pareto distribution of the first type with parameters b, a.

(6) If we set $h(X) = [1 + X]^{-a}, a > 0, d = 1, c_i = \theta_i, \in \{1,2\}, \alpha = 0$ and $\beta = \infty$, we obtain characterizations concerning a mixture of two exponentiated Pareto distributions of the second type with respective parameters a, θ_1 and a, θ_2. For $\theta_1 = \theta_2 = \theta$, we have characterizations concerning the second type exponentiated Pareto distribution.

(7) If we set $h(X) = e^{-\gamma X}, \gamma > 0, d = 1, c_i = \theta_i, i\epsilon\{1,2\}, \alpha = 0$ and $\beta = \infty$, we obtain characterizations concerning a mixture of two generalized exponential distributions with respective positive parameters $\gamma, \theta_1$and γ, θ_2. For $\theta_1 = \theta_2 = \theta$, we have characterizations concerning the generalized exponential distribution with parameters γ and θ. If in addition, we set, θ=1, we obtain characterizations concerning the ordinary exponential distribution with parameter γ.

(8) If we set $h(X) = \frac{-Z(X)}{Z(\beta)-g(n)/[1-m(n)]}$, $d = \frac{-g(n)/[1-m(n)]}{Z(\beta)-g(n)/[1-m(n)]}$, $c_1 = c_2 = \frac{m(n)}{1-m(n)}$, where $m(\cdot)$ and $g(\cdot)$ are finite real valued functions of n and $Z(x)$ is a differentiable function defined on (α, β) such that $\lim_{x\to\alpha^+} Z(x) = \frac{g(n)}{1-m(n)}$ and $\lim_{x\to\beta^-} Z(x) = Z(\beta)$, we obtain characterizations concerning a class of distributions defined by Talwalker [20].

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