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Natural Vibrations of Sound Waves in Pipes

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Abstract

The purpose of this paper is to construct the asymptotic for natural frequencies of the sound problem using the method of Wentzel-Kramers-Brillouin (WKB) and find the secular equation for eigenvalues of a sound wave in a pipe closed on the left and open on the right.

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1 Introduction

The sound wave is one of the elements most used. Therefore, the vibration of sound waves has been studied extensively and continues to receive attention

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in the current literature. The wave equation for a vibrating sound wave can be found in [4, 6]. General linearized wave equations are obtained by Pierce and Goldstein [1, 5]. For the equation of the sound with a uniform small parameter, you can use the method (WKB) Wentzel - Kramers - Brillouin, also known in the literature as the approximation Liouville - Green. In (Akulenko, Nesterov [2], 2005) WKB method was used for finding asymptotic high frequency, this method is to obtain asymptotic series for solutions powers with a small parameter.

Although the WKB method is developed for differential equations of n-th order (Fedoryuk [3], 1993), this technique can be applied to sound waves in a pipe.

In the next section one takes the equation, for uniform sound waves in a pipe.

2 Mathematical formulation

In this section we find the approximate solution as a linear combination of two linearly independent solutions. After imposing the boundary conditions for the case of sound waves in a pipe closed on the left and open on the right, which is reduced to a system of two equations with two unknowns which has nontrivial solutions if its determinant is zero. And finally we obtain the asymptotic determinant equation when $\epsilon \to 0$. Then we have eigenvalues.

2.1 Sound equation

The sound equation for longitudinal vibrations in the uniform sound approximation have been described by (Feynman [4], 1969)

$$\rho \frac{\partial^2 s}{\partial t^2} = B \frac{\partial^2 s}{\partial x^2}, \tag{1}$$

where B is the bulk modulus of sound, ρ is the density, and s is the longitudinal displacement. It is assumed that $0 \le x \le l$ where l is the length of the pipe.

Looking for the solution of the equation of sound (1) in the form

$$y(x,t) = e^{i\omega t} v(x), \tag{2}$$

where w is the frequency. Substituting (2) in the equation (1) we obtain second order ordinary differential equation for $v(x)$

$$-\omega^2 \rho v(x) = Bv''(x). \tag{3}$$

Taking the boundary conditions for sound waves on both sides. Equation (3) is transformed into the following problem:

$$\begin{aligned} -\frac{B}{\rho} v''(x) &= \omega^2 v(x) \\ v(0) = v'(l) &= 0. \end{aligned} \tag{4}$$

The discrete spectrum of the problem (4) constitutes a sequence w_n of real numbers tending to infinity when $n \to \infty$. In view of the above can be consider $w^2 = \frac{1}{\epsilon^2}$, where $\epsilon \to 0$ (Akulenko, Nesterov [2], 2005). Therefore the problem of sound waves, (4) becomes

$$-\frac{B}{\rho} v''(x) = \frac{1}{\epsilon^2} v(x), \tag{5}$$

$$v(0) = v'(l) = 0. \tag{6}$$

In the next section we calculate the secular equation of the problem (5)–(6).

3 Main result

This section states and solves the problem of sound waves at high frequency, which is applied the WKB method. The mathematical formulation of the problem is the searching for nontrivial solutions of the problem (5)–(6).

The main result is as follows

Theorem 3.1. *The eigenvalues of the problem* (5)*–(6) are given by* $w_n^2 = \frac{1}{\varepsilon_n^2}$, *with*

$$\omega_n = \frac{(n - \frac{1}{2})\pi}{\sqrt{\frac{\rho}{B}} l} \left(1 + O\left(\frac{1}{n}\right)\right), \quad n = 1, \ldots, \quad n \to \infty,$$

where ε_n *is the solution of the secular equation* (21).

Proof. Given that $\omega^2 = \frac{1}{\varepsilon^2}$ then equation (5) can be transformed into

$$v(x) = -\frac{\varepsilon^2 B}{\rho} v''(x). \tag{7}$$

Following the traditional WKB method, the analytical solution approximates equation (5) can be replaced by a power series given by the following

$$v(x) = A(x,\varepsilon) e^{\frac{i\phi(x)}{\varepsilon}}, \quad \varepsilon \to 0, \tag{8}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \dots, \quad \varepsilon \to 0, \tag{9}$$

with $\phi(x)$ and $A_j(x)$, $j = 0, 1, 2, \dots$ are smooth functions and unknown. Replacing (8) and each of the derivatives of $v(x)$ in (7), we have the following expression

$$\begin{aligned} A(x,\varepsilon) = & \frac{B}{\rho}[-2\varepsilon A_x(x,\varepsilon) + \phi_x^2(x) A(x,\varepsilon) - 2i\varepsilon\phi_x(x) A_x(x,\varepsilon) \\ & - i\varepsilon\phi_{xx}(x) A(x,\varepsilon)] + O(\varepsilon^2), \quad \varepsilon \to 0. \end{aligned} \tag{10}$$

Replacing (9) on both sides of (10) we obtain

$$\begin{aligned} & A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \cdots = \frac{B}{\rho}\big[\phi_x^2(x)(A_0(x) + \varepsilon A_1(x) \\ & + \varepsilon^2 A_2(x) + \cdots) - 2i\varepsilon\phi_x(x)(A_{0x}(x) + \varepsilon A_{1x}(x) + \varepsilon^2 A_{2x}(x) + \cdots) \\ & - i\varepsilon\phi_{xx}(x)\big(A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \cdots\big)\big] + O(\varepsilon^2) \\ = & \frac{B}{\rho}\phi_x^2(x) A_0(x) + \varepsilon\left[\frac{B}{\rho}\phi_x^2(x) A_1(x) \right. \\ & \left. - \frac{2iB}{\rho}\phi_x(x) A_{0x}(x) - \frac{iB}{\rho}\phi_{xx}(x) A_0(x)\right] \\ & + O(\varepsilon^2), \quad \varepsilon \to 0. \end{aligned} \tag{11}$$

Equating the coefficients of the asymptotic series in ε and taking corresponding to ε^0 in (11) and using that $A_0 \neq 0$ as seen in the equation (15), it can be obtained

$$\varepsilon^0 : \frac{B}{\rho}\phi_x^2(x) = 1. \tag{12}$$

From equation (12) and choosing the corresponding equality to ε^1 in (11), we obtain

$$\varepsilon^1 : 2\phi_x(x)A_{0x}(x) + \phi_{xx}(x)A_0(x) = 0. \tag{13}$$

Equating the asymptotic series are obtained more equations but we will consider only the first two equations, other equations are of order $O(\varepsilon^2)$. Since equation (12) has two real roots with opposite signs, we obtain

$$\phi_k(x) = (-1)^k \sqrt{\frac{\rho}{B}}x, \quad k = 1, 2. \tag{14}$$

From the equation (13) and separating the functions $A_0(x)$, $\phi_x(x)$ and integrating on both sides, it follows that

$$A_0(x) = C(\phi_x(x))^{-\frac{1}{2}}, \tag{15}$$

where C is a non-zero arbitrary constant. Therefore, differentiating with respect to x in the equation (14) and substituting (15), function $A_0(x)$ can be expressed as follows

$$A_0(x) = C. \tag{16}$$

Therefore, replacing (14) and (16) in (8) which is the solution $v(x)$ of (5), we have

$$v_1(x) = d_1 \sin\left(\frac{\sqrt{\frac{\rho}{B}}x}{\varepsilon}\right) + O(\varepsilon), \quad \varepsilon \to 0, \tag{17}$$

$$v_2(x) = d_2 \cos\left(\frac{\sqrt{\frac{\rho}{B}}x}{\varepsilon}\right) + O(\varepsilon), \quad \varepsilon \to 0, \tag{18}$$

and writing the solution (5) as the first term of the linear combination of $v_1(x), v_2(x)$

$$v(x) = c_1 \sin\left(\frac{\sqrt{\frac{\rho}{B}}x}{\varepsilon}\right) + c_2 \cos\left(\frac{\sqrt{\frac{\rho}{B}}x}{\varepsilon}\right), \tag{19}$$

$\varepsilon \to 0$, and c_1, c_2 are constants. It is noted that $\sin\left(\frac{x}{\varepsilon}\right), \cos\left(\frac{x}{\varepsilon}\right)$ are linearly independent.

The boundary conditions for a sound wave in a pipe closed on the left and open on the right are given by

$$v(0) = v'(l) = 0. \tag{20}$$

From the solution (19) and boundary conditions (20), yields a homogeneous system of two equations for two constants $c_i, i = 1, 2$. This system has non-trivial solutions when

$$\begin{vmatrix} 0 & 1 \\ \cos\left(\frac{\sqrt{\frac{\rho}{B}}l}{\varepsilon}\right) & \sin\left(\frac{\sqrt{\frac{\rho}{B}}l}{\varepsilon}\right) \end{vmatrix} = 0. \tag{21}$$

The equation (21) is the secular equation for natural frequency $\omega_n = \varepsilon_n^{-1}$. Therefore

$$\cos\left(\frac{\sqrt{\frac{\rho}{B}}l}{\varepsilon}\right) = 0, \quad \varepsilon \to 0, \quad \varepsilon = \varepsilon_n = \frac{\sqrt{\frac{\rho}{B}}l}{(n-\frac{1}{2})\pi} \tag{22}$$

then

$$\omega_n = \frac{(n-\frac{1}{2})\pi}{\sqrt{\frac{\rho}{B}}l}\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, n = 1, \ldots.$$

From (22) and initial conditions, the eigenfunction is

$$v_n(x) = c_3 \mathrm{sen}\left(\frac{\sqrt{\frac{\rho}{B}}x}{\varepsilon_n}\right),$$

where c_3 is an arbitrary constant. □

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