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Discreteness of the solutions to equations of mathematical physics

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Abstract

The equations of mathematical physics, which describe actual processes, are defined on manifolds (tangent or others) that are not integrable. The solutions of equations on such manifolds cannot be functions since the derivatives of such solutions do not made up a differential. The exact solutions (functions), which are possible only on integrable structures, can be realized only discretely under the realization of additional conditions. The process of realization of discrete solutions reveals the mechanism of generation of integrable structures, which format integrable manifolds, and emergence of physical structures, which made up physical fields and an occurrence of waves, eddies and so on.

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1 Introduction

The equations of mathematical physics, which describe physical processes are defined on nonintegrable manifolds (for example, on such as tangent mani-

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fold). As is shown in present paper, such equations have solutions of two types: the solutions that are not functions (depend on the integration path) and so called generalized solutions, which are functions but are realized discretely. The discreteness of generalized solutions relates to the fact that such solutions are obtained only under realizations of additional conditions, which define integral structures. The realization of additional conditions (which are caused by some degrees of freedom) proceeds with the help of degenerate transformation that executes the transition from original nonintegrable manifold to the integrable structure being realized.

It should be noted that the results of present paper were obtained with the help of skew-symmetric differential forms. In this case, in addition to closed exterior forms, which possess the invariant properties, it has been used the skew-symmetric differential forms, which are obtained from differential equations. These skew-symmetric forms are evolutionary ones. Such evolutionary forms contain an unconventional mathematical apparatus that includes such basic concepts as degenerate transformations and nonidentical relations and can generate closed exterior forms.

In the second section of present paper the properties of solutions to the equations of mathematical physics, which describe any processes, are studied by the example of the first-order partial differential equations, and the conjugacy of derivatives with respect to different variables is analysed. In this case the discreteness of solutions is due to the fact that the derivatives of the function desired appear to be nonconjugated on the tangent manifold.

In the next section the set of differential equations of mechanics and physics of material systems (continuous media) is considered. This is a set of equations that describe the conservation laws for material systems (the conservation laws for energy, momentum, angular momentum, and mass). From the equations analysed it follows the nonidentical relation in skew-symmetric forms that just describes the process of realization of generalized solutions. In this case, the discreteness of solutions depends on the consistence of the equations and on the properties of conservation laws. It is shown that the process of discrete realization of generalized solutions has an unique mathematical and physical sense. Such a process, firstly, discloses the mechanism of generation of integrable structures and integrable manifolds and, secondly, explains the mechanism of emergence of physical structures and advent formations in material systems like vortices, waves and so on.

In the last section it is shown that the peculiarities of solutions to the equations of conservation laws for material systems disclose the connection between the field-theory equations and the equations for material systems and demonstrate the connection between physical fields and material systems.

2 Specific features of solutions to equations describing actual processes; Main Results

Specific features of solutions to differential equations describing physical processes can be demonstrated by the example of first-order partial differential equation using the properties of skew-symmetric differential forms [1],[2].

Let us take the simplest case: the first-order partial differential equation

$$F(x^i,\, u,\, p_i) = 0, \quad p_i \,=\, \partial u/\partial x^i \tag{1}$$

Let us consider the functional relation

$$du \,=\, \theta \tag{2}$$

where $\theta \,=\, p_i\, dx^i$ is a skew-symmetric differential form of the first degree (the summation over repeated indices is implied).

By constructing the skew-symmetric form $\theta \,=\, p_i\, dx^i$ must be a differential since p_i are derivatives of the function desired.

However, in general case when differential equation (1) describes any physical processes, this differential form appears to be an unclosed form, and for this reason it is not a differential. The differential form $\theta \,=\, p_i\, dx^i$ appears to be an unclosed form because its differential is nonzero. Really, the differential $d\theta$ is equal to $K_{ij}dx^idx^j$, where $K_{ij} = \partial p_j/\partial x^i - \partial p_i/\partial x^j$ are components of the differential form commutator. From equation (1) it does not follow (explicitly) that the derivatives $p_i \,=\, \partial u/\partial x^i$, which obey to the equation (and to given boundary or initial conditions), are consistent, that is, their mixed derivatives are commutative. Components of commutator K_{ij} is nonzero. Therefore, the differential form commutator and the differential of form θ are nonzero.

In the general case, when differential equation (1) describes any physical processes, the functional relation (2) is nonidentical one. The left-hand side of this relation involves a differential, and the right-hand side includes the differential form $\theta = p_i\,dx^i$, which is not a differential.

The nonidentity of functional relation (2) means that equation (1) is non-integrable: the derivatives p_i of equation do not made up a differential. The solution u of equation (1) obtained from such derivatives is not a function of only variables x^i. This solution will depend on the commutator K_{ij}.

To obtain a solution that is a function (i.e., the derivatives of this solution made up a differential), it is necessary to add the closure condition for the form $\theta = p_i dx^i$ and for the relevant dual form (in the present case the functional F plays a role of a form dual to θ) [1]:

$$\begin{cases} dF(x^i,\, u,\, p_i) = 0 \\ d(p_i\,dx^i) = 0 \end{cases} \tag{3}$$

If we expand the differentials, we get a set of homogeneous equations with respect to dx^i and dp_i (in the $2n$-dimensional space):

$$\begin{cases} \left(\dfrac{\partial F}{\partial x^i} + \dfrac{\partial F}{\partial u}\,p_i\right)\,dx^i + \dfrac{\partial F}{\partial p_i}\,dp_i = 0 \\ dp_i\,dx^i - dx^i\,dp_i = 0 \end{cases} \tag{4}$$

It is well known that *vanishing the determinant* composed of coefficients at dx^i, dp_i is a solvability condition of the system of homogeneous differential equations. This leads to relations:

$$\frac{dx^i}{\partial F/\partial p_i} = \frac{-dp_i}{\partial F/\partial x^i + p_i \partial F/\partial u} \tag{5}$$

Relations (5) specify the integrating direction, which defines an integrable structure, that is, a pseudostructure, on which the form $\theta = p_i\,dx^i$ turns out to be closed one, i.e. it becomes a differential, and the identical relation follows from relation (2). On the pseudostructure, which is defined by relation (5), the derivatives of differential equation (1) constitute a differential $\delta u = p_i dx^i = du$ (on the pseudostructure), and this means that the solution to equation (1) becomes a function. Solutions, namely, functions on the pseudostructures, are so-called generalized solutions. The characteristics, characteristic surfaces, singular points, potential surfaces, and others are examples of pseudostructures or their formations.

One can see that the solutions, which are functions, are obtained only under additional condition. This additional condition, as one can see, is *vanishing the determinant*. Such an additional condition is a condition of degenerate transformation. The degenerate transformation executes a transition from tangent nonintegrable manifold of differential equations to integral structures (pseudostructures) or the surfaces of cotangent manifold. (The Legendre transformations are examples of such a transformation.)

It should be underlined the following.

The derivatives of differential equation are defined on tangent manifold, that are not integrable. As it has been shown above, such derivatives do not made up a differential. The solution that had been obtained from such derivatives is not a function. This solution is defined on tangent manifold. And the solution that is a function (generalized solution) is defined on integrable structures or surfaces, which belong to cotangent integrable manifold.

Thus one obtains that the transition from the solution, which is not a function, to generalized solution, which is a function, is realized as a transition from tangent nonintegrable manifold to integrable structures of cotangent manifold. This means that the solutions, which are functions (i.e. generalized solutions) are realizes under additional conditions (the conditions of degenerate transforms), are discrete solutions. Such solutions have discontinuities in the direction normal to pseudostructures (to integrable structures or surfaces).

The first-order partial differential equation has been analyzed, and the functional relation with the form of the first degree has been considered.

Similar functional properties have all differential equations describing actual processes. And, if the order of differential equation is k, the functional relation with the k-degree form corresponds to this equation.

Thus one can see that the solutions to equations of mathematical physics, on which no additional conditions are imposed, are not functions. They depend on a quantity that relates to nonconjugacy of differential equation derivatives (this quantity is described by the commutator of unclosed skew-symmetric form made up of differential equation derivatives). The solutions that are functions (generalized solutions) are realized only under additional requirements, namely, the conditions of degenerate transformations, and hence, they are discrete solutions (defined only on pseudostructures) and have discontinuities in the direction normal to pseudostructures.

[Investigation of nonidentical functional relations lies at the basis of the qualitative theory of differential equations. It is well known that the qualitative theory of differential equations is based on the analysis of unstable solutions and the integrability conditions. From the functional relation it follows that the dependence of the solution on the commutator leads to instability, and the closure conditions of skew-symmetric forms constructed by derivatives are integrability conditions. That is, the qualitative theory of differential equations, which solves the problem of unstable solutions and integrability, bases on the properties of nonidentical functional relation.]

3 Properties of solutions for equations of mechanics and physics of continuous medium

While studying the solution of partial differential equations, the conjugacy of derivatives with respect to different variables was analyzed. When describing physical processes in continuous media (in material systems) one obtains not one differential equation but a set of differential equations. And in this case it is necessary to investigate the conjugacy of not only derivatives with respect to different variables but also the conjugacy (consistency) of the equations of this set. In this case, from this set of equations one also obtains a nonidentical relation that enables one to investigate the integrability of equations and the specific features of their solutions.

(In particular, the consistency of equations analysed in paper [3]. In that paper the consistency conditions were referred to as dynamical conditions.)

The equations of mechanics and physics of continuous media (of material systems) is a set of equations that describe the conservation laws for energy, linear momentum, angular momentum, and mass. The Euler and Navier-Stokes equations are examples of such a set of equations [4].

Let us analyze the consistency of the equations for energy and linear momentum.

In the accompanying reference system, which is tied to the manifold made up by the trajectories of particles (elements of material system), the equation for energy is written in the form

$$\frac{\partial\psi}{\partial\xi^1} = A_1 \tag{6}$$

Here ξ^1 are the coordinates along the trajectory, ψ is the functional of the state that specifies material system, A_1 is the quantity that depends on specific features of the material system and on external (with respect to the local domain) energy actions onto the system. {The action functional, entropy and wave function can be regarded as examples of the functional ψ [5]. Thus, the equation for energy expressed in terms of the action functional S has following form: $DS/Dt = L$, where $\psi = S$ and $A_1 = L$ is the Lagrange function. The equation for energy of ideal gas can be presented in the form: $Ds/Dt = 0$, where s is entropy [4].}

In a similar manner, in the accompanying reference system the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial\psi}{\partial\xi^\nu} = A_\nu, \quad \nu = 2, ... \tag{7}$$

where ξ^ν are the coordinates in the direction normal to the trajectory, A_ν are the quantities that depend on the specific features of material system and on external force actions.

Eqs. (6) and (7) can be convoluted into the relation

$$d\psi = A_\mu\, d\xi^\mu, \quad (\mu = 1, \nu) \tag{8}$$

where $d\psi$ is the differential expression $d\psi = (\partial\psi/\partial\xi^\mu)d\xi^\mu$.

Relation (8) can be written as

$$d\psi = \omega \tag{9}$$

here $\omega = A_\mu\, d\xi^\mu$ is the skew-symmetric differential form of the first degree. [It should be noted that skew-symmetric differential forms, which are obtained from differential equations, are not exterior skew-symmetric forms, because, in contrast to exterior forms, they are defined on tangent or accompanying manifolds, which are not integrable. Such skew-symmetric differential forms are evolutionary ones since they are obtained from evolutionary equations [2]. Below it will be shown the properties of such evolutionary forms that enable to study specific features of solutions to the equations of mathematical physics.] (A concrete form of relation (9) and its properties in the case of the Euler and Navier-Stokes equations were considered in papers [6, 7]. In this case the functional ψ is the entropy s.)

Relation (9) has been obtained from the equation of conservation laws for energy and linear momentum. In this relation the form ω is that of the first degree. Taking into account the equations of conservation laws for angular momentum and mass, the evolutionary relation can be written as

$$d\psi \,=\, \omega^p \tag{10}$$

where the form degree p takes the values $p \,=\, 0, 1, 2, 3$. (A concrete form of relation (10) for $p = 2$ were considered for electromagnetic field in paper [2] (Appendix 3) and in paper http://arxiv.org/pdf/math-ph/0310050v1.pdf

The relation obtained possesses the properties that enable one to investigate the integrability of original set of equations and the properties of its solutions.

This relation is, firstly, an evolutionary one since the original equations are evolutionary.

Secondly, it, as well as functional relation (2), turns out to be nonidentical.

To justify this, we shall analyze the relation (9).

The evolutionary relation $d\psi \,=\, \omega$ is a nonidentical relation as it involves the unclosed skew-symmetric differential form $\omega \,=\, A_\mu d\xi^\mu$. The commutator of the form ω is nonzero. The components of commutator of such a form can be written as follows:

$$K_{\alpha\beta} \,=\, \left(\frac{\partial A_\beta}{\partial \xi^\alpha} \,-\, \frac{\partial A_\alpha}{\partial \xi^\beta}\right)$$

Such a commutator cannot vanish since the coefficients A_μ depend on external energetic and force actions, which are not potential and consistent. This means that the differential of the form ω is nonzero as well. Thus, the form ω proves to be unclosed and cannot be a differential.

[The commutator of the form ω describes a force that causes the deformation of accompanying manifold, on which the form ω is defined. In this case an additional term in the commutator, namely, the commutator of unclosed metric form of manifold, appears in the process of manifold deformation.]

The nonzero value of the commutator, and accordingly, of the form differential, means that the differential form ω in the right-hand side of evolutionary relation is unclosed form and cannot be a differential like left-hand side of this relation. This points out to the fact that the evolutionary relation proves to be nonidentical. (Nonidentical relation had been noted in paper [8]. In this case, a possibility to make use a sign of equality of this relation was allowed.)

[Since the evolutionary relation has been obtained from the conservation law equations, a nonidentity of this relation is a result of noncommutativity of conservation laws [7]. The noncommutativity of conservation laws, and, as a result, a nonidentity of evolutionary relation, is a reason of discreteness of the solutions to mathematical physics equations.]

The nonidentity of the evolutionary relation means that the initial equations of conservation laws are not consistent, and hence, they are not integrable.

Without realization of additional conditions the solutions to these equations will depend on the evolutionary form commutator and hence cannot be functions.

The equation can became integrable only under realization of additional conditions, when the evolutionary form commutator and the form differential vanish. These are just conditions of degenerate transformation when from the evolutionary form (with nonzero differential) one obtains closed exterior form (with vanishing differential). Under degenerate transformation, from nonidentical relation the relation that is identical on pseudostructure is obtained, and this points out to consistency of equations and to local (on pseudostructure) integrability of original equations. In this case the solutions to equations are generalized solutions, which are functions, but realized discretely (only under additional conditions).

The questions of how the conditions of degenerate transformation are realized and how the degenerate transformation proceeds arise.

The conditions of degenerate transformation relate to degrees of freedom of material system. Realization of these conditions proceeds while varying evolutionary relation, which appears to be selfvarying one. (An evolutionary relation contains two objects one of which appears to be unmeasurable and cannot be compared with another one, and therefore the process of mutual variation cannot terminate). The selfvariation of evolutionary relation leads to realization of the conditions of degenerate transformation.

If the conditions of degenerate transformation are realized, from the unclosed evolutionary form with nonvanishing differential $d\omega^p \neq 0$ one can obtain a differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$d\omega^p \neq 0 \to$ (degenerate transformation) $\to d_\pi \omega^p = 0$, $d_\pi{}^*\omega^p = 0$

where the conditions $d_\pi \omega^p = 0$ and $d_\pi{}^*\omega^p = 0$ are conditions of closuring the exterior and dual forms.

The condition $d_\pi{}^*\omega^p = 0$ is an equation of a certain pseudostructure π, on which the differential of evolutionary form vanishes: $d_\pi \omega^p = 0$. That is, the closed (inexact) exterior form ω_π^p is obtained on pseudostructure.

On the pseudostructure from evolutionary relation $d\psi = \omega^p$ it is obtained the identical relation $d_\pi \psi = \omega_\pi^p$, since the closed exterior form ω_π^p is a differential of some differential form. (The relation obtained will be identical one as the left- and right-hand sides of the relation contain differentials).

The identity of the relation obtained from the evolutionary relation means that on pseudostructures the original equations for material system (the equations of conservation laws) become consistent and integrable. This points to the fact that **pseudostructure is an integrable structure**.

Pseudostructures constitute the integrable surfaces (such as characteristics, singular points, potentials of simple and double layers, and others) on which the quantities of material system desired (such as the temperature, pressure, density) become functions of only independent variables and do not depend on the commutator (and on the path of integrating). This are generalized solutions. They may be found by means of integrating the equations of conservation laws for material systems.

Since generalized solutions are defined only on realized integrable structures (pseudostructures), they or their derivatives have discontinuities in the direction normal to integrable structure [9].

One can see that the integrable surfaces are obtained from the condition of degenerate transformation of the evolutionary relation. The conditions of degenerate transformation are a vanishing of such functional expressions as determinants, Jacobians, Poisson's brackets, residues, and others. They are connected with the symmetries, which can be due to the degrees of freedom of the material systems under consideration (for example, the translational, rotational and oscillatory degrees of freedom of material system).

The degenerate transformation is realized as a transition from the noninertial frame of reference to the locally inertial system, i.e. a transition from nonintegrable manifold (for example, tangent or accompanying) to integrable structures and surfaces.

Thus, one can see that the solutions to the set of equations, as well as in the case of a single equation, may be of two types: the solutions that are not functions, that is, they depend not only on independent variables, and generalized solutions, which are functions, and are obtained onto under realization of additional conditions (which determine integrable structures or surfaces). Since generalized solutions are obtained only under realization of additional conditions, they are discrete solutions. (Solutions to the Euler and Navier-Stokes equations are examples of such solutions [10]).

Under the realization of additional conditions, which determine integrable structures, it is carried out a transition from the solutions, which are not functions, to generalized solutions (functions). This transition proceeds as a transition from tangent (accompanying) manifold (on which are defined the solutions that are not functions) to integrable structures, on which generalized solutions are defined.

3.1 Mathematical and physical meaning of discrete realization of the solutions to equations of mathematical physics

The discrete realization of generalized solution to equations of mathematical physics relates to a realization of dual form, which describes a pseudostructure, and this points to realization of integrable structure. It appears that integrable structures and integrable manifolds are generated by the equations of mathematical physics that describes physical processes in material systems.

The realization of dual form and associated closed exterior form is a realization of differential-geometrical structure, namely, a pseudostructure with conservative quantity. (Below it will be shown that such structures, which can be called as "physical structures", have a physical meaning. Structures that form physical fields and associated pseudometric and metric manifolds are just such structures [11].)

On the other hand, a realization of differential-geometrical structure means that the integrable structure (pseudostructure) is realized and it is obtained a function that corresponds to generalized solution. Such a function or its derivative have a discontinuity in the direction normal to integrable structure. Realization of integrable structure with such a discontinuous function and transition from nonintegrable tangent manifold to integrable structure describes emergence of a certain formation. Waves, vortices, fluctuations, turbulent pulsations and so on are examples of such formations [6,7,12].

Thus we obtain that the discrete realization of generalized solution points to an occurrence of a differential-geometrical structure (physical structure), and, on other hand, an emergence of a certain (observable) formation in material system that is described by generalized solution. It will be shown below that such a duality has a physical meaning, namely, it discloses a relation of physical fields and material systems and enables one to understand peculiarities of field-theory equations.

4 Connection between the field-theory equations and the equations for material systems

A connection between the field-theory equations and the equations for material systems bases on the properties of conservation laws.

4.1 Conservation laws

The field-theory equations, as well as the equations for material systems, are connected with the properties of conservation laws. However, conservation laws for physical fields, which are described by field-theory equations, and conservation laws for material systems have a different meaning. The conservation laws for material systems are conservation laws for energy, linear momentum, angular momentum, and mass that establish the balance between the change of physical quantities and the external action. They are described by differential (or integral) equations.

In field theory "conservation laws" are those that claim an existence of conservative physical quantities or objects. Such conservation laws are described by closed exterior skew-symmetric forms. (The Noether theorem is an example.) These are conservation laws for physical fields.

Since the conservation laws for physical fields and the conservation laws for material systems have a different meaning, the equations for material systems and the field-theory equations are equations of different type.

It appears that the field-theory equations, unlike the equations for material systems, have a form of relations. This is due to the fact that the solutions to field theory equations must be differentials (rather then derivatives) since the closed exterior forms, i.e. differentials, are assigned to conservation laws for physical fields.

And jet, as it will be shown, there exist a connection between the field-theory equations and those for material systems [13]. The connection between the field-theory equations and equations for material systems points to the fact that there exists a correspondence between field-theory equations and evolutionary relation obtained from the equations for material systems.

4.2 Correspondence between field-theory equations and evolutionary relation

The nonidentical evolutionary relation is a relation for functionals such as wave-function, action functional, entropy, and others [5]. The field-theory equations are those for such functionals.

Another correspondence between the field-theory equations and the non-identical evolutionary relation relates to the fact that all field-theory equations have the form of relations. They can be relations in differential forms or in the forms of their tensor or differential (i.e. expressed in terms of derivatives) analogs.

The Einstein equation is a relation in differential forms.

The Dirac equation relates Dirac's *bra*- and *cket*- vectors, which made up a differential form of zero degree.

The Maxwell equations have the form of tensor relations.

The Schrődinger's equations have the form of relations expressed in terms of derivatives and their analogs.

The fact that from the field-theory equations, as well as from the nonidentical evolutionary relation, the identical relation, which contains the closed exterior form, is obtained also points to a correspondence between the non-identical evolutionary relations and the field-theory equations.

The closed exterior forms or their tensor or differential analogs, which are obtained from identical relations, are solutions to the field-theory equations.

As one can see, from the field-theory equations it follows such identical relation as

(1) the Poincare invariant, which connects closed exterior forms of first degree;

(2) the relations $d\theta^2 = 0$, $d^*\theta^2 = 0$ are those for closed exterior forms of second degree obtained from Maxwell equations;

(3) the Bianchi identity for gravitational field.

From the Einstein equation it is obtained the identical relation in the case when the covariant derivative of the energy-momentum tensor vanishes.

Thus one can see that there exists a correspondence between the field-theory equations and evolutionary relation obtained from the equations for material systems.

Such a correlation between the field-theory equations and nonidentical relation points to the fact that the equations of field theory are connected with the equations for material systems. It has been shown in paper [2, 13, 14] that such a connection enables one to understand basic principles of field theory.

The connection between field-theory equations, which are based on conservation laws for physical fields, and the equations for material systems, that are conservation law equations for material systems, points to the fact that there exists a connection between conservation laws for material systems and conservation laws for physical fields.

One can see that physical structures (on which the conservation laws for physical fields are fulfilled) are those from which physical fields and relevant manifolds are formatted.

5 Conclusion

The equations of mathematical physics, which describe physical processes, are defined on tangent nonintegrable manifold. In present paper it has been shown that such equations have the solutions of two types: the solution that are not functions (depend on the integration path) and generalized solutions, which are functions but are realized discretely. The realization of generalized solution relates to the realization of integral structure (pseudostructure). The integrable manifolds with generalized solutions (such as characteristic manifolds, potential surfaces, and others) are formatted by these integrable structures. It occurs that integrable manifolds are generated by the equations of mathematical physics that describe physical processes proceeded in material systems.

The discrete realization of generalized solutions has an unique physical meaning.

The process of realization of generalized solution reveals the processes that proceed in material systems, namely, a generation of physical structures and an emergence of any observable formations in material system.

It appears that physical structures, of which physical fields and pseudometric or metric manifolds are formatted, are just such physical structures. This demonstrates that physical fields are generated by material systems and points out to a connection between the equations of field theory and the equations for material systems.

And the formations emerged in material systems, such as fluctuations, waves, turbulent pulsations, are formations that are described by discrete generalized solutions.

The properties of generalized solutions and the process of their realization discloses a role of conservation laws in evolutionary processes.

It should be emphasized that the results presented have been obtained due to the apparatus of skew-symmetric differential forms. In this case in addition to closed exterior forms, which possess the invariant properties, it has been used the skew-symmetric differential forms. These skew-symmetric forms are obtained from differential equations and are evolutionary ones. Such evolutionary skew-symmetric forms possess a specific feature, namely, they can generate closed exterior forms. This enables one to describe the discrete transitions, the processes of conjugating various operators and the mechanisms of evolutionary processes.

References

[1] E. Cartan, *Les Systemes Differentials Exterieus ef Leurs Application Geometriques*, Paris, Hermann, 1945.

[2] L.I. Petrova, *Exterior and evolutionary differential forms in mathematical physics: Theory and Applications*, Lulu.com, 2008.

[3] V.I. Smirnov, A course of higher mathematics, Moscow, *Tech. Theor. Lit.* **4**, (1957), in Russian.

[4] J.F. Clarke and M. Machesney, *The Dynamics of Real Gases*, Butterworths, London, 1964.

[5] L.I. Petrova, Physical meaning and a duality of concepts of wave function, action functional, entropy, the Pointing vector, the Einstein tensor, *Journal of Mathematics Research*, **4**(3), (2012).

[6] L.I. Petrova, Integrability and the properties of solutions to Euler and Navier-Stokes equations, *Journal of Mathematics Research*, **4**(3), (2012).

[7] L.I. Petrova, The noncommutativity of the conservation laws: Mechanism of origination of vorticity and turbulence, *International Journal of Theoretical and Mathematical Physics*, **2**(4), (2012).

[8] J.L. Synge, Tensorial methods in dynamics, Department of Applied Mathematics University of Toronto, *Applied Mathematics Series*, (2), 1936.

[9] L.I. Petrova, Relationships between discontinuities of derivatives on characteristics and trajectories, *J. Computational Mathematics and Modeling*, **20**(4), (2009), 367-372.

[10] L.I. Petrova, Discreteness of solutions to Euler and Navier-Stokes equations, *Abstracts of the International Conference on Differential Equations and Dynamical Systems*, Suzdal, (2012), 224-225.

[11] L.I. Petrova, Mechanism of formatting the pseudometric and metric manifolds, *Proceedings of the conference Modern problems in mathematics and mechanics*, Moscow State University, Mathematics, **4**(3), (2011), 124-130.

[12] L.I. Petrova, The mechanism of generation of physical structures, *Proceedings of 18th International Symposium on Nonlinear Acoustics, Stockholm*, American Institute of Physics (AIP), New York, (2008), 151-154.

[13] L.I. Petrova, The connection of field-theory equations with the equations for material systems, *Proceedings of International Meeting Physical Interpretation of Relativity Theory*, Moscow, (2007), 140-147.

[14] L.I. Petrova, Foundations of unified and general field theories, *International Journal of Theoretical and Mathematical Physics*, **2**(6), (2012).