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On the maximum of randomly weighted sums with subexponential tails

Saliou Diouf[1] **and Aliou Diop**[2]

Abstract

Consider the randomly weighted sums $S_n(\theta) = \sum_{k=1}^{n} \theta_k X_k$, where $\{X_k, 1 \leq k \leq n\}$ is a sequence of independent real-valued random variables with common subexponential distribution function F, and let $\{\theta_k, 1 \leq k \leq n\}$ a sequence of positive random variables, independent of $\{X_k, 1 \leq k \leq n\}$ and satisfying $a \leq \theta_k \leq b$ for some $0 < a \leq b < \infty$ for all $1 \leq k \leq n$. Under a suitable summability condition on the upper endpoints of $|\theta_k|$ we prove that

$$\mathbb{P}\left(\max_{1\leq n<\infty} \sum_{k=1}^{n} \theta_k X_k > x\right) \sim \sum_{k=1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right).$$

This result appears as a direct extension of the results obtained in [12].

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[1] LERSTAD, Université Gaston Berger, BP. 234 Saint-Louis, Sénégal.
[2] LERSTAD, Université Gaston Berger, BP. 234 Saint-Louis, Sénégal.

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1 Introduction

Throughout this paper we are interested in the tail probability of randomly weighted sums $S_n(\theta), n = 1, 2\cdots$ defined by (1) and their maximum $M_n(\theta)$ defined by (2)

$$S_n(\theta) = \sum_{k=1}^{n} \theta_k X_k, \tag{1}$$

$$M_n(\theta) = \max_{1\leq m\leq n} S_m(\theta), \tag{2}$$

$$M_\infty(\theta) = \max_{1\leq m<\infty} S_m(\theta), \tag{3}$$

where $\{X_k, 1 \leq k \leq n\}$ is a sequence of independent, identically distributed, and real-valued random variables with common distribution function F, its tail is denoted by $\bar{F} = 1 - F$, and satisfies the tail balancing condition,

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{\mathbb{P}(|X| > x)} = p \quad , \quad \lim_{x\to\infty} \frac{\bar{F}(-x)}{\mathbb{P}(|X| > x)} = 1 - p. \tag{4}$$

Let $\{\theta_k, 1 \leq k \leq n\}$ a sequence of positive random variables, independent of $\{X_k, 1 \leq k \leq n\}$, we consider that each weight θ_k has upper endpoint

$$c_k = c(\theta_k) = sup\{c : \mathbb{P}(\theta_k < c) \leq 1, k = 1, 2, \cdots\}$$

and we assume that for some $\delta > 0$, and for $m \geq 1$

$$\sum_{k=m+1}^{\infty} c_k^{1-\delta} < 1. \tag{5}$$

A sequence of random variables $\{\Gamma_k, 1 \leq k \leq n\}$ is bounded

1. of type I if $\mathbb{P}(a \leq \Gamma_k \leq b) = 1$ holds for some $0 < a \leq b < \infty$ and all $1 \leq k \leq n$,

2. of type II if $\mathbb{P}(0 < \Gamma_k \leq b) = 1$ holds for some $0 < b < \infty$ and all $1 \leq k \leq n$,

3. of type III if $\mathbb{P}(a \leq \Gamma_k < b) = 1$ holds for some $0 < a < \infty$ and all $1 \leq k \leq n$.

For more details see [12].

A distribution function F or its corresponding random variable X is said to be heavy tailed to the right if $\mathbb{E}\exp(rX) = \infty$ for $r > 0$. A necessary condition for F to be heavy tailed is that $\bar{F}(x) > 0$ for any real number x.
The most important class of heavy-tailed distribution functions is the subexponential class denoted $\mathcal{S}$.
A distribution function F supported on $[0, \infty)$ belongs to the class $\mathcal{S}$ if

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\bar{F}(x)} = n, \quad \text{for} \quad n \geq 2, \tag{6}$$

where $\overline{F^{*n}}$ denote the n-fold convolution of F.
A closely related class is the $\mathcal{L}$ of long-tailed distributions.
A distribution function F on $(-\infty, +\infty)$ belongs to the class $\mathcal{L}$ if

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\bar{F}(x)} = 1, \tag{7}$$

holds for some (or, equivalently,) $y \geq 0$.
It is known that

$$\mathcal{S} \subset \mathcal{L}. \tag{8}$$

Another closely related class is the class $\mathcal{D}$ of distribution functions with dominated variations. By definition, a distribution function F belongs to the class $\mathcal{D}$ if

$$\lim_{x\to\infty} \sup \frac{\bar{F}(xy)}{\bar{F}(x)} < \infty, \tag{9}$$

holds for any (or, equivalent) $0 < y < 1$.
It is well known that

$$\mathcal{R}_{-\alpha} \subset \mathrm{ERV}(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}. \tag{10}$$

where $\mathcal{R}_{-\alpha}$ denote the Regular Variation Class, $\mathrm{ERV}(-\alpha, -\beta)$ the Extended Regular Variation Class, $\mathcal{C}$ the Consistent Variation Class.
For more details about heavy-tailed distribution and their application see [1] or [4]. The weighted sums plays an important role in actuarial and economic study; see [5],[12],[16].
Following the works of [10], [11] and [13], we consider this model :

$$S_n = \sum_{k=1}^{n} X_k \prod_{j=1}^{k} Y_j, \tag{11}$$

where $\{X_n, n \geq 1\}$ is as in model (1), and $\{Y_n, n \geq 1\}$ is another sequence of nonnegative random variables distributed on $[0, \infty)$, the two sequences being mutually independent.
In economics, the random variable X_n in the model (11) is the total loss during period n and the random variable Y_n is the discount factor from time n to time $n-1, n = 1, 2, \cdots$. Thus, the sum S_n represents the aggregated discounted losses by time n of an insurer in a stochastic economic environment.
For the model (11) let

$$M_n = \max_{0 \leq k \leq n} \sum_{k=1}^{n} X_k \prod_{j=1}^{k} Y_j \tag{12}$$

and

$$M_\infty = \max_{0 \leq k < \infty} \sum_{k=1}^{n} X_k \prod_{j=1}^{k} Y_j, \tag{13}$$

The quantity M_n defined by (12) describes the maximum of the discounted losses of the insurer by time n, n = 1, 2, . . ., and the quantity M_∞ defined by (13) describes the ultimate maximum of the discounted losses.
For the model (11), the finite and infinite time ruin probabilities are defined for an insurer whose initial wealth is $x \geq 0$ as

$$\psi(x, n) = \mathbb{P}(M_n > x)$$

and

$$\psi(x) = \mathbb{P}(M_\infty > x),$$

respectively.
We notice that the model (11) reduces to model (1) with $\theta_k = \prod_{j=1}^{k} Y_j$, a product of positive random variables.
The motivation of this work comes from the fact that the main result is an extension of [12], and can play an important role in various applied and theoretical problems. For example, in the field of the economic or of the insurance, our result can be used to evaluate the probability of the ultimate maximum of the discounted losses.
Throughout this article, all limit relationships are for $x \to \infty$ unless otherwise stated. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \sim b(x)$ if $\lim a(x) / b(x) = 1$.

Our objective in this work is to establish the following result

$$\mathbb{P}\left(\max_{1\leq n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right) \sim \sum_{k=1}^{\infty}\mathbb{P}\left(\theta_k X_k > x\right). \quad (14)$$

If $F \in \mathcal{S}$ and $\{\theta_k, 1 \leq k \leq n\}$ is bounded of type I then, (15) proved that the relation

$$\mathbb{P}\left(\max_{1\leq n<m}\sum_{k=1}^{n}\theta_k X_k > x\right) \sim \sum_{k=1}^{m}\mathbb{P}\left(\theta_k X_k > x\right) \quad (15)$$

hold as $x \to \infty$. It is important to note that the passage from (15) to (14) is not obvious.

The rest of this paper is organized as follows. Section 2 presents the main results. Section 3 proposes an application of the main result for a thresholds models.

2 Main Results

Recall the randomly weighted sums (1), and their maximum defined by

$$\max_{1\leq n<\infty}\sum_{k=1}^{n}\theta_k X_k,$$

where $\{X_k, 1 \leq k \leq n\}$ is a sequence of independent, identically distributed (i.i.d), and real-valued random variables with common distribution function F and $\{\theta_k, 1 \leq k \leq n\}$ be another sequence of positive random variables.We suppose $\{X_k, 1 \leq k \leq n\}$, $\{\theta_k, 1 \leq k \leq n\}$ are mutually independent and

$$\sum_{k=1}^{\infty}\mathbb{P}\left(\theta_k X_k > x\right) < \infty. \quad (16)$$

The main result of this paper is the following:

Theorem 2.1. *Consider the randomly weighted sums (1) and their maximum (2). We suppose $F \in \mathcal{S}$ and satisfies the balancing condition (4), the upper endpoint of $\{\theta_k, 1 \leq k \leq n\}$ verifies (5). If $\{\theta_k, 1 \leq k \leq n\}$ is bounded of type I then*

$$\mathbb{P}\left(\max_{1\leq n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right) \sim \sum_{k=1}^{\infty}\mathbb{P}\left(\theta_k X_k > x\right) \quad (17)$$

hold as $x \to \infty$.

Corollary 2.2. *If $F \in \mathcal{L} \cap \mathcal{D}$ and $\{\theta_k, 1 \leq k \leq n\}$ is bounded of type II, then the (17) is hold.*

Proof

If $F \in \mathcal{L} \cap \mathcal{D}$ and $\{\theta_k, 1 \leq k \leq n\}$ is bounded of type II, then by [12] the relation (19) is hold and the rest being identical to the proof of theorem the (2.1). ■

Proof of theorem We have for any $m \geq 1$

$$\mathbb{P}\left(\max_{1\leq n<\infty} \sum_{k=1}^{n} \theta_k X_k > x\right) \geq \mathbb{P}\left(\max_{1\leq n<m} \sum_{k=1}^{n} \theta_k X_k > x\right). \tag{18}$$

By [12], we have

$$\mathbb{P}\left(\max_{1\leq n<m} \sum_{k=1}^{n} \theta_k X_k > x\right) \sim \sum_{k=1}^{m} \mathbb{P}\left(\theta_k X_k > x\right). \tag{19}$$

Trivially

$$\sum_{k=1}^{m} \mathbb{P}\left(\theta_k X_k > x\right) \sim \sum_{k=1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right) - \sum_{k=m+1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right) \tag{20}$$

Then, from (16) we deduce that $\sum_{k=1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right)$ converge and $\sum_{k=m+1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right)$ is asymptotically negligible in comparison to $\sum_{k=1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right)$. Combining (18), (19) and (20) we have

$$\mathbb{P}\left(\max_{1\leq n<\infty} \sum_{k=1}^{\infty} \theta_k X_k > x\right) \geq \sum_{k=1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right). \tag{21}$$

We will complete the proof if we show

$$\mathbb{P}\left(\max_{1\leq n<\infty} \sum_{k=1}^{\infty} \theta_k X_k > x\right) \leq \sum_{k=1}^{\infty} \mathbb{P}\left(\theta_k X_k > x\right). \tag{22}$$

Note that as in [15], for any $m \geq 1$

$$\max_{1\leq n<\infty} \sum_{k=1}^{n} \theta_k X_k \leq \max_{1\leq n\leq m} \sum_{k=1}^{n} \theta_k X_k + \sum_{k=m+1}^{\infty} \theta_k X_k^{+}.$$

Then for any choose of ℓ such that $0 < \ell < 1$ and $x \geq 0$, we have

$$\begin{aligned} \mathbb{P}\left(\max_{1\leq n<\infty} \sum_{k=1}^{n} \theta_k X_k > x\right) &\leq \\ \leq \mathbb{P}\left(\max_{1\leq n\leq m} \sum_{k=1}^{n} \theta_k X_k > (1-\ell)x\right) + \mathbb{P}\left(\sum_{k=m+1}^{\infty} \theta_k X_k^{+} > \ell x\right) & \\ &= A_m + B_m. \end{aligned}$$

By [12], we have

$$A_m = \mathbb{P}\left(\max_{1\leq n\leq m}\sum_{k=1}^{n}\theta_k X_k > (1-\ell)x\right) \sim \sum_{k=1}^{m}\mathbb{P}\left(\theta_k X_k > (1-\ell)x\right). \qquad (23)$$

Now we show that B_m is asymptotically negligible if $\sum_{j>m} c_j^{1-\delta} < 1$.
Here we are going to use the fact that, for the generic r.v. X and the common distribution function F of $\{X_i, i \geq 1\}$, we have $\mathbb{P}(X^+ \leq x) = 1 - \mathbb{P}(X^+ > x) = 1 - \mathbb{P}(X > x) = 1 - \bar{F}(x)$.
Hence

$$\begin{aligned} B_m &\leq \mathbb{P}\left[\sum_{k=m+1}^{\infty}\theta_k X_k^+ > \sum_{k=m+1}^{\infty} c_k^{1-\delta}\ell x\right] \\ &\leq \mathbb{P}\left(\bigcup_{k=m+1}^{\infty}\left[\theta_k X_k^+ > c_k^{1-\delta}\ell x\right]\right) \\ &\leq \sum_{k=m+1}^{\infty}\mathbb{P}(\theta_k X_k^+ > \ell c_k^{1-\delta}x) \\ &\leq \sum_{k=m+1}^{\infty}\mathbb{P}(\theta_k X_k > \ell c_k^{1-\delta}x). \end{aligned}$$

By (16) we have

$$\sum_{k=m+1}^{\infty}\mathbb{P}(\theta_k X_k > \ell c_k^{1-\delta}x) < \varepsilon.$$

Then B_m is asymptotically negligible and

$$\mathbb{P}\left(\max_{1\leq n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right) \leq \sum_{k=1}^{m}\mathbb{P}\left(\theta_k X_k > (1-\ell)x\right). \qquad (24)$$

Let $m \to \infty$ and $\ell \to 0$ in (24) we have

$$\mathbb{P}\left(\max_{1\leq n<\infty}\sum_{k=1}^{n}\theta_k X_k > x\right) \leq \sum_{k=1}^{\infty}\mathbb{P}\left(\theta_k X_k > x\right). \qquad (25)$$

Combining (21) and (25) we obtain (17). ∎

3 Application

Let the following threshold model with subexponential innovations

$$\alpha_t = \begin{cases} \phi_1\alpha_{t-1} + \varepsilon_t^{(1)}, & \text{if} \quad Y_t > \tau, \\ \phi_2\alpha_{t-1} + \varepsilon_t^{(2)}, & \text{if} \quad Y_t \leq \tau, \end{cases} \tag{26}$$

where τ and ϕ_i are non random constants and with threshold variable Y_t. The sequences $\{\varepsilon_t^i, i = 1, 2\}$ are sequence of iid random variables with common distribution function F.

When we define $I_{1t} = \mathbf{1}_{\{Y_{t-\delta} > \tau\}}$, $I_{2t} = 1 - I_{1t}$ and $q = \mathbb{P}(Y_t \leq \tau)$ the model (26) can be written as

$$\alpha_t = \phi_{(t)}\alpha_{t-1} + \varepsilon_t \tag{27}$$

where

$$\phi_{(t)} = \phi_1 I_{1t} + \phi_2 I_{2t} \qquad \textit{and} \qquad \varepsilon_t = \varepsilon_t^{(1)} I_{1t} + \varepsilon_t^{(2)} I_{2t}.$$

The equation (27) is a stochastic difference equation where the pairs $(\phi_{(t)},\ \varepsilon_t)_t$ are sequences of independent and not identically distributed $\mathbb{R}^2$-valued random variables.

We may give an financial example of model (26) introduced by Breidt [3] for a financial return Y_t defined by :

$$Y_t = \sigma \exp\left(\frac{\alpha_t}{2}\right) \varepsilon_t. \tag{28}$$

Where α_t is an open-loop threshold autoregressive process defined by 26 with $\tau = 0$ (see [14]).

The model (26) is called a threshold autoregressive stochastic volatility model (TARSV). The log-volatility process $(\alpha_t)_t$ has a piecewise linear structure. It switches between two first-order autoregressive process according to the sign of the previous return. In this framework, σ is positive constant and $(\varepsilon_t)_t$ is a sequence of independent and identically distributed random variables with zero mean and its variance is taken to be one. When either $\mid \phi_1 \mid= 1$ and $\mid \phi_2 \mid\neq 1$ or $\mid \phi_1 \mid\neq 1$ and $\mid \phi_2 \mid= 1$, the process defined in (26) is stationary in some regimes and mildly explosive in others. These models are stationary in some regimes and mildly explosive in others. See Gonzalo and Montesinos [7]. Gouriroux and Robert [8] studied the ACR(1) process where there is a switching between white noise and a random walk.

Now assume the following conditions holds :

- $\mathbf{H_1}$: $(\varepsilon_t^{(i)})_t$ is a sequence of independent, identically distributed (i.i.d) random variables ($i = 1, 2$) and satisfied the following condition :

$$\mathbb{E}[\log^+ \varepsilon_1^{(i)}] < +\infty, \tag{29}$$

where $\log^+ x = \max(0, \log x)$.

- $\mathbf{H_2}$: For each $i = 1, 2$, the two sequences of random variables $(\varepsilon_t^{(i)})_t$ and $(Y_t)_t$ are independent and $(\varepsilon_t^{(1)})_t$ and $(\varepsilon_t^{(2)})_t$ are independent.

- $\mathbf{H_3}$: The sequence of independent and identically distributed random variables $(\varepsilon_t^{(i)})_t$ whose common distribution F is subexponential and satisfies the follow tail balancing condition :

$$\lim_{x\to\infty} \frac{\mathbb{P}(\varepsilon_1^{(i)} > x)}{\mathbb{P}(|\varepsilon_1^{(i)}| > x)} = p, \quad \lim_{x\to\infty} \frac{\mathbb{P}(\varepsilon_1^{(i)} < -x)}{\mathbb{P}(|\varepsilon_1^{(i)}| > x)} = 1 - p. \tag{30}$$

The next proposition gives the strict stationarity of the process α_t defined by (27). The result follows from Theorem 1 of Brandt [2].

Proposition 3.1. *(strict stationarity) Assume H_1 and H_2 and suppose that $\phi_1^q \phi_2^{1-q} < 1$. Then, for all $t \in \mathbb{Z}$ the series α_t defined by (27) admits the following expansion*

$$\alpha_t = \sum_{j=0}^{\infty} \left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) \varepsilon_{t-j}. \tag{31}$$

Now let us consider $\Theta_j = \prod_{k=0}^{j-1} \phi_{(t-j)}$ and we assume that the sequences $\{\varepsilon_j, j \geq 1\}$ and $\{\Theta_j, j \geq 1\}$ satisfy corresponding conditions imposed in Theorem (2.1). Then by applying Theorem (2.1) we obtain immediately the following asymptotics in next Theorem.

Theorem 3.2.

- *If $F \in \mathcal{S}$ and $\{\Theta_k, 1 \leq k \leq n\}$ is bounded of type I, or*

- *If $F \in \mathcal{L} \cap \mathcal{D}$ and $\{\Theta_k, 1 \leq k \leq n\}$ is bounded of type II*

then we have

$$\mathbb{P}\left(\max_{1\leq n<\infty} \sum_{j=0}^{n} \left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) \varepsilon_{t-j} > x \right) \sim \sum_{j=1}^{\infty} \mathbb{P}\left(\left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) \varepsilon_{t-j} > x \right) \tag{32}$$

hold as $x \to \infty$.

Proof We have only to show

$$\sum_{j=1}^{\infty} \mathbb{P}\left(\left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) \varepsilon_{t-j} > x \right) < \infty \tag{33}$$

By [6] we have

$$\mathbb{P}\left(Y_j > x\right) \sim \bar{F}(x)\beta_j, \tag{34}$$

where

$$Y_j = \left(\left(\prod_{k=0}^{j-1} \phi_{(t-k)} \right) \varepsilon_{t-j} \right) \tag{35}$$

and

$$\beta_j = \begin{cases} q^j & \text{if} \quad \phi_1 = 1 \qquad \mid \phi_2 \mid < 1, \\ (p\delta_j + (1-p)\delta_{j+1})p^{-1}q^j & \text{if} \quad \phi_1 = -1 \qquad \mid \phi_2 \mid < 1, \\ (1-q)^j & \text{si} \quad \phi_2 = 1 \qquad \mid \phi_1 \mid < 1, \\ (p\delta_j + (1-p)\delta_{j+1})p^{-1}(1-q)^j & \text{if} \quad \phi_2 = -1 \qquad \mid \phi_1 \mid < 1, \\ 0 & \text{if} \quad |\phi_1| < 1, \qquad |\phi_2| < 1. \end{cases} \tag{36}$$

with

$$\delta_j = \begin{cases} 1 & \text{if k even} \\ 0 & \text{if k old.} \end{cases} \tag{37}$$

Combining (34) and (36) we have (33).

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