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Finite Integrals Involving Jacobi Polynomials and I-function

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Abstract

The aim of the present paper is to evaluate new finite integral formulas involving Jacobi polynomials and I-function. These integral formulas are unified in nature and act as key formula from which we can obtain as their special cases. For the sake of illustration we record here some special cases of our main formulas which are also new. The formulas establish here are basic in nature and are likely to find useful applications in the field of science and engineering.

Mathematics Subject Classification : 33C45, 33C60
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1 Introduction

The I-function will be defined and represented as follows [1, p. 26, Eqn.(2.1.41)]:

$$I^{m,n}_{p_i,q_i,r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \phi(\xi)z^{\xi}d\xi \qquad (1)$$

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where

$$\phi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j\xi)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=1}^{q_i}\Gamma(1 - b_{ji} + \beta_{ji}\xi)\prod_{j=1}^{p}\Gamma(a_{ji} - \alpha_{ji}\xi)\right]} \tag{2}$$

and m, n, p_i, q_i are integers satisfy $0 \leq n \leq p_i, 1 \leq m \leq q_i (i = 1...r)$ r is finite $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are positive integer and a_j, b_j, a_{ji}, b_{ji} are complex numbers. I-function which is a generalized form of the well known H-function [2, p.10, Eqn.(2.1.1)] In the sequel the I-function will be studied under the following conditions of existence:

$$A_i > 0, |\arg z| < \frac{A_i\pi}{2} \tag{3}$$

where

$$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji}, \forall i = (1, 2, ..., r) \tag{4}$$

The general class of polynomials $S_{n_1,...,n_r}^{m_1,...,m_r}[x]$ introduced by Srivastava will be defined and represented as follows [3, p.185, Eqn.(7)]:

$$S_{n_1,...,n_r}^{m_1,...,m_r}[x] = \sum_{l_1=0}^{[n_1/m_1]} ... \sum_{l_r=0}^{[n_r/m_r]} \prod_{i=1}^{r} \frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} x^{l_i} \tag{5}$$

where $n_1, ..., n_r = 0, 1, 2, ...; m_1, ..., m_r$ is an arbitrary positive integers, the coefficients A_{n_i,l_i} $(n_i, l_i \geq 0)$ are arbitray constants, real or complex. On suitably specializing the coefficients $A_{n_i,l_i}, S_{n_1...n_r}^{m_1...m_r}[x]$ yeilds a number of known polynomials as its special cases. These includes, among other, the Bessel Polynomials, the Lagurre Polynomials, the Hermite Polynomials, the Jacobi Polynomials, the Gould-Hopper Polynomials, the Brafman Polynomials and several others [4,p.158-161]

The following known results [5, p.945, Eqn.(16)] and [6, p.172, Eqn.(29)] for the Jacobi Polynomials $P_n^{(\alpha,\beta)}[x]$ [7, p.254, Eqn.(1)], will be required in our investigation.

$$P_\mu^{(\alpha,\beta)}(t+y)P_\mu^{(\alpha,\beta)}(t-y) = \frac{(-1)^\mu(1+\alpha)_\mu(1+\beta)_\mu}{(\mu!)^2} \times \sum_{n=0}^{\mu} \frac{(-\mu)_n(1+\alpha+\beta+\mu)_n}{(1+\alpha)_n(1+\beta)_n} P_n^{(\alpha,\beta)}(x)t^n \tag{6}$$

$$\frac{1}{y}(1-t+y)^{-\alpha}(1-t+y)^{-\beta} = 2^{-\alpha-\beta}\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n \tag{7}$$

where y denotes $(1-2xt+t^2)^{1/2}$ in both (6) and (7).

2 Main Integrals

We establish the following integrals:
First Integral

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}S_{n_1\ldots n_{r'}}^{m_1\ldots m_{r'}}[w(1-x)^u(1+x)^v] \times I_{p_i,q_i,r}^{m,n}\left[z(1-x)^h(1+x)^k\middle|_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}}^{(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}\right]dx$$
$$= 2^{\rho+\sigma-1}\sum_{l_1=0}^{[n_1/m_1]}\cdots\sum_{l_{r'}=0}^{[n_{r'}/m_{r'}]}\prod_{i=1}^{r'}\frac{(-n_i)_{m_il_i}}{l_i!}A_{n_i,l_i}w^{l_i}2^{(u+v)l_i} \times I_{p_i+2,q_i+1;r}^{m,n+2}\left[z2^{(h+k)}\middle|_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i};(1-\rho-\sigma-(u+v)l_i,h+k)}^{(1-\rho-ul_i,h);(1-\sigma-vl_i,k);(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}\right] \tag{8}$$

The equation (8) will be converge under the conditions given in equation (3) and

I. $\rho \geq 1, \sigma \geq 1; u \geq 0, v \geq 0; h \geq 0, k \geq 0$($h$ and k are not both zero simultaneously)

II. $\mathrm{Re}(\rho) + h\min\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right] > 0$

III. $\mathrm{Re}(\sigma) + k\min\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right] > 0$

Proof: To establish the integral (8), we express the I-function occurring in its left-hand side in terms of Mellin-Barnes contour integral given by equation (1), the integral class of polynomial occurring therein the series form given by equation (5) and the interchange the order of summations and integration and the order of x-and ξ-integrals (which is permissible under the conditions stated with equation (8) and evaluating the integral with the help of a modified form of the formula [8, p. 314, en.(3)],we easily arrive at the first integral after a little simplification.

Second Integral:

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_{\mu}^{\alpha,\beta}(t+y)P_{\mu}^{\alpha,\beta}(t-y)\times$$

$$I_{p_i,q_i,r}^{m,n}\left[z(1-x)^h(1+x)^k\Big|_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}}^{(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}\right]dx$$

$$= 2^{\rho+\sigma-1}\frac{(-1)^{\mu}\Gamma(1+\alpha+\mu)\Gamma(1+\beta+\mu)}{(\mu!)^2}\times$$

$$\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-\mu)_{n'}(1+\alpha+\beta+\mu)_{n'}}{\Gamma(1+\alpha+n')\Gamma(1+\beta+n')}\frac{(-n')_l}{l!}\begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$$I_{p_i+2,q_i+1;r}^{m,n+2}\left[z2^{(h+k)}\Big|_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i};(1-\rho-\sigma-l,h+k)}^{(1-\rho-l,h);(1-\sigma,k);(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}\right] \qquad (9)$$

where $y = (1-2xt+t^2)^{1/2}$, the conditions of the above result can be easily obtained from those of first integral.

Third Integral

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}\frac{1}{y}(1-t+y)^{-\alpha}(1+t+y)^{-\beta}\times$$

$$I^{m,n}_{p_i,q_i,r}\left[z(1-x)^h(1+x)^k\Big|^{(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}}\right]dx$$

$$= 2^{-\alpha-\beta+\rho+\sigma-1}\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-n')_l}{l!}\begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$$I^{m,n+2}_{p_i+2,q_i+1;r}\left[z2^{(h+k)}\Big|^{(1-\rho-l,h);(1-\sigma,k);(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i};(1-\rho-\sigma-l,h+k)}\right] \tag{10}$$

where $y=(1-2xt+t^2)^{1/2}$, the conditions of the above result can be easily obtained from the first integral.

To establish equation (9) and (10) the following result is required, which the is special case of first integral:

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P^{\alpha,\beta}_{n'}(x)I^{m,n}_{p_i,q_i,r}\left[z(1-x)^h(1+x)^k\Big|^{(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}}\right]dx$$

$$= 2^{\rho+\sigma-1}\sum_{l=0}^{n'}\frac{(-n')_l}{l!}\begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$$I^{m,n+2}_{p_i+2,q_i+1;r}\left[z2^{(h+k)}\Big|^{(1-\rho-l,h);(1-\sigma,k);(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i};(1-\rho-\sigma-l,h+k)}\right] \tag{11}$$

The conditions of the above result can be easily obtained from those of first integral.

If we put $m_1=...=m_{r'}=1;n_1=...=n_{r'}=n';w=\frac{1}{2};u=1,v=0$ and $A(n_1,l_1,...,n_{r'},l_{r'}) = \begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}$ in equation (8) then the polynomial $S^1_{n'}\left[\frac{1-x}{2}\right]$occuring therein breaks up into the Jacobi Polynomials $P^{(\alpha,\beta)}_{n'}[x]$ [9, p.68, eq.(4.3.2)] and the equation (8) reduces to the equation (11) after a little simplification.

Proof of second integral: Put the value of $P^{(\alpha,\beta)}_{\mu}(t+y)P^{(\alpha,\beta)}_{\mu}(t-y)$ from equation(6) to the left hand side of equation (9) and interchanging the order of integration and summation, then using the equation (11), we easily arrive at the required second integral after little simplification.

Proof of third integral: Put the value of $\frac{1}{y}(1-t+y)^{-\alpha}(1+t+y)^{-\beta}$ from equation (7) to the left hand side of equation (10) and interchanging the order of integration and summation, then using the equation (11), we easily arrive at the required third integral after little simplification.

3 Special Cases of Main Integrals

(a) If we put $r=1, \alpha_j=\beta_j=\alpha_{ji}=\beta_{ji}=1$ then I-function reduces to the general type of G-function $I^{m,n}_{p_i,q_i,r}\left[z\left|^{(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}}_{(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}}\right.\right]=G^{m,n}_{p,q}\left[z\left|^{(a_j,1)_{1,n};(a_j,1)_{n+1,p}}_{(b_j,1)_{1,m};(b_j,1)_{m+1,q}}\right.\right]$, the equation (9) and (10) takes place in the following form:

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_{\mu}^{\alpha,\beta}(t+y)P_{\mu}^{\alpha,\beta}(t-y)\times$$
$$G^{m,n}_{p,q}\left[z(1-x)^{h}(1+x)^{k}\left|^{(a_j,1)_{1,n};(a_j,1)_{n+1,p}}_{(b_j,1)_{1,m};(b_j,1)_{m+1,q}}\right.\right]dx$$
$$=$$
$$2^{\rho+\sigma-1}\frac{(-1)^{\mu}\Gamma(1+\alpha+\mu)\Gamma(1+\beta+\mu)}{(\mu!)^2}\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-\mu)_{n'}(1+\alpha+\beta+\mu)_{n'}}{\Gamma(1+\alpha+n')\Gamma(1+\beta+n')}\times$$
$$\frac{(-n')_l}{l!}\begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$$G^{m,n+2}_{p+2,q+1}\left[z2^{(h+k)}\left|^{(1-\rho-l,h);(1-\sigma,k);(a_j,1)_{1,n};(a_j,1)_{n+1,p}}_{(b_j,1)_{1,m};(b_j,1)_{m+1,q};(1-\rho-\sigma-l,h+k)}\right.\right] \tag{12}$$

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}\frac{1}{y}(1-t+y)^{-\alpha}(1+t+y)^{-\beta}\times$$
$$G^{m,n}_{p,q}\left[z(1-x)^{h}(1+x)^{k}\left|^{(a_j,1)_{1,n};(a_j,1)_{n+1,p}}_{(b_j,1)_{1,m};(b_j,1)_{m+1,q}}\right.\right]dx$$
$$=2^{-\alpha-\beta+\rho+\sigma-1}\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-n')_l}{l!}\begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$$G^{m,n+2}_{p+2,q+1}\left[z2^{(h+k)}\Big|^{(1-\rho-l,h);(1-\sigma,k);(a_j,1)_{1,n};(a_j,1)_{n+1,p}}_{(b_j,1)_{1,m};(b_j,1)_{m+1,q};(1-\rho-\sigma-l,h+k)}\right] \quad (13)$$

The conditions of convergence of the above equation (12) and (13) can be obtained from those of the first integral.

(b) If we put $r=1, m=1, n=p_i=p, q_i=q+1, b_1=0, \beta_1=1, a_j=1-a_j, b_{ji}=1-b_j, \beta_{ji}=\beta_j$ then I-function reduces to Wright's generalized Hypergeometric function, i.e. $I^{1,p}_{p,q+1;1}\left[z\ \Big|^{(1-a_j,\alpha_j)_{1,p}}_{(0,1),(1-b_j,\beta_j)_{1,q}}\right] = {}_p\psi_q\left[\begin{array}{c}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array};-z\right]$ then the equation (9) and (10) reduces to the following form:

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P^{\alpha,\beta}_{\mu}(t+y)P^{\alpha,\beta}_{\mu}(t-y)\times$$
$${}_p\psi_q\left[\begin{array}{c}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array};-z(1-x)^h(1+x)^k\right]dx$$
$$=$$
$$2^{\rho+\sigma-1}\frac{(-1)^{\mu}\Gamma(1+\alpha+\mu)\Gamma(1+\beta+\mu)}{(\mu!)^2}\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-\mu)_{n'}(1+\alpha+\beta+\mu)_{n'}}{\Gamma(1+\alpha+n')\Gamma(1+\beta+n')}\times$$
$$\frac{(-n')_l}{l!}\left(\begin{array}{c}n'+\alpha\\n'\end{array}\right)\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$${}_{p+2}\psi_{q+1}\left[\begin{array}{c}(1-\rho-l,h);(1-\sigma,k);(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q};(1-\rho-\sigma-l,h+k)\end{array};-z2^{(h+k)}\right] \quad (14)$$

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}\frac{1}{y}(1-t+y)^{-\alpha}(1+t+y)^{-\beta}\times$$
$${}_p\psi_q\left[\begin{array}{c}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{array};-z(1-x)^h(1+x)^k\right]dx$$
$$=2^{-\alpha-\beta+\rho+\sigma-1}\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-n')_l}{l!}\left(\begin{array}{c}n'+\alpha\\n'\end{array}\right)\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$${}_{p+2}\psi_{q+1}\left[\begin{array}{c}(1-\rho-l,h);(1-\sigma,k);(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q};(1-\rho-\sigma-l,h+k)\end{array};-z2^{(h+k)}\right] \quad (15)$$

The conditions of convergence of the above equation (14) and (15) can be obtained from those of the first.

(c) If we put $r=1, m=1, n=p_1=p, q_i=q, b_1=0, \beta_1=1, a_j=1-a_j, b_{ji}=b_j, \alpha_j=\beta_j=\alpha_{ji}=\beta_{ji}=1$, I-function reduces to the generalized hypergeometric function,i.e $I^{1,p}_{p,q+1}\left[z\Big|^{(1-a_j)_{1,p}}_{(0,1);(1-b_j)1,q}\right]=\frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}{}_pF_q\left[{}^{a_1,\ldots,a_p;}_{b_1,\ldots,b_q;}-z\right]$, then the equation (9) and (10) takes the following form:

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}P_\mu^{\alpha,\beta}(t+y)P_\mu^{\alpha,\beta}(t-y){}_pF_q\left[{}^{a_1,\ldots,a_p;}_{b_1,\ldots,b_q;}-z(1-x)^h(1+x)\right]dx$$

$$=$$

$$2^{\rho+\sigma-1}\frac{(-1)^\mu\Gamma(1+\alpha+\mu)\Gamma(1+\beta+\mu)}{(\mu!)^2}\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-\mu)_{n'}(1+\alpha+\beta+\mu)_{n'}}{\Gamma(1+\alpha+n')\Gamma(1+\beta+n')}\times$$

$$\frac{(-n')_l}{l!}\begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$$\frac{\Gamma(\rho+l)\Gamma(\sigma)}{\Gamma(\rho+\sigma+l)}{}_pF_q\left[{}^{(1-\rho-l,h);(1-\sigma,k);a_1,\ldots,a_p;}_{b_1,\ldots,b_q;(1-\rho-\sigma-l,h+k);}-z2^{h+k}\right] \tag{16}$$

$$\int_{-1}^{1}(1-x)^{\rho-1}(1+x)^{\sigma-1}\frac{1}{y}(1-t+y)^{-\alpha}(1+t+y)^{-\beta}\times$$

$${}_pF_q\left[{}^{a_1,\ldots,a_p;}_{b_1,\ldots,b_q;}-z(1-x)^h(1+x)\right]dx$$

$$=2^{-\alpha-\beta+\rho+\sigma-1}\sum_{n'=0}^{\mu}\sum_{l=0}^{n'}\frac{(-n')_l}{l!}\begin{pmatrix} n'+\alpha \\ n' \end{pmatrix}\frac{(\alpha+\beta+n'+1)_l}{(\alpha+1)_l}\times$$

$$\frac{\Gamma(\rho+l)\Gamma(\sigma)}{\Gamma(\rho+\sigma+l)}{}_pF_q\left[{}^{(1-\rho-l,h);(1-\sigma,k);a_1,\ldots,a_p;}_{b_1,\ldots,b_q;(1-\rho-\sigma-l,h+k);}-z2^{h+k}\right] \tag{17}$$

The conditions of convergence of the above equation (16) and (17) can be obtained from those of the first integral.

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