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Calculating the Values of Multiple Integrals by Replacing the Integrands Interpolation by Interpolation Polynomial

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Abstract

The work deals with the construction of multidimensional quadrature formulas for computation of the multiple integrals' values by replacing the integrands by interpolation polynomial. We've proved the quadrature formulas' correctness.

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1 Introduction

In [1-4] we'll consider the methods of constructing cubature formulas on the

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basis of the method of repeated application of different, Simpson and Chebyshev. In this paper, to construct cubature formulas for the n-fold integrals values calculating, we used method of replacing of the integrands by interpolating polynomial. For this purpose, let's replace the $f(x_1, x_2, ...x_n)$ integrand by the following polynomial:

$$f(x_1, x_2,..., x_n) = L(x_1, x_2,..., x_n) = f(x_1^{(1)}, x_2^{(1)},..., x_n^{(1)})P_1(x_1, x_2,..., x_n) +$$
$$+ f(x_1^{(2)}, x_2^{(2)},..., x_n^{(2)})P_2(x_1, x_2,..., x_n) + ... +$$
$$+ f(x_1^{(n)}, x_2^{(n)},..., x_n^{(n)})P_n(x_1, x_2,..., x_n).$$

Integrating (1) ,we have:

$$J \equiv \iint_{(\Omega)} ... \int f(x_1, x_2,..., x_n) dx_1 dx_2 ... dx_n \approx \iint_{(\Omega)} ... \int L(x_1, x_2,..., x_n) dx_1 dx_2 ... dx_n \approx$$
$$= \sum_{i=1}^{n} f(x_1^{(i)}, x_2^{(i)},..., x_n^{(i)}) \iint_{(\Omega)} ... \int P_i(x_1, x_2,..., x_n) dx_1 dx_2 ... dx_n =$$
$$= \sum_{i=1}^{n} C_i f(x_1^{(i)}, x_2^{(i)},..., x_n^{(i)}), \qquad (2)$$

where

$$C_i = \iint_{(D)} \int P_i(x_1, x_2,..., x_n) dx_1 dx_2 ... dx_n.$$

In simple fields, the C_i values calculation is not difficult because $P_i(x_1, x_2,..., x_n)$ is a polynomial. In complex areas to calculate the C_i values it is appropriate to use the V.L. Rvachev's R-functions [2,5]. This method determines the points of integration.

For simplicity, let's consider the n-dimensional rectangular region and some grid lines on it, formed by:

$$x_{i_1} = a_1 + i_1 h_1$$
$$.....................\qquad (3)$$
$$x_{i_n} = a_n + i_n h_n$$

$$i_1 = \overline{0, n_1};\ i_2 = \overline{0, n_2}; ...\ i_n = \overline{0, n_n};\ h_i = \frac{h - q}{n_i}, i = \overline{1, n}.$$

In this case we use the $(x_{i_1}, x_{i_2,...,} x_{i_n})$: interpolation formula, using the nodes:

$$f(x_1, x_2, ..., x_n) = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}...\sum_{i_n=0}^{n_n} f(x_{i_1}, x_{i_2}, ..., x_{i_n}) \frac{w_{n_1}(x_1)w_{n_2}(x_2)...w_{n_n}(x_n)}{(x_1 - x_{i_1})(x_2 - x_{i_2})...(x_n - x_{i_n})w'_{n_1}(x_{i_1})...w'_n(x_{i_n})} + R \qquad (4)$$

where

$$\begin{aligned} w_{n_1}(x_1) &= (x_1 - x_1^{(1)})(x_1 - x_2^{(1)})...(x_1 - x_n^{(1)}), \\ w_{n_2}(x_2) &= (x_2 - x_1^{(2)})(x_2 - x_2^{(2)})...(x_2 - x_n^{(2)}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w_{n_n}(x_n) &= (x_n - x_1^{(n)})(x_n - x_2^{(n)})...(x_n - x_{n_n}^{(n)}) \end{aligned} \qquad (5)$$

Integrating it, we obtain

$$J = \iint_{(\Omega)}...\int f(x_1, x_2, ..., x_n)dx_1dx_2...dx_n = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}...\sum_{i_n=0}^{n_n} f(x_{i_1}, x_{i_2}, ..., x_{i_n}) \times$$
$$\times \int_{a_1}^{A_1} \frac{w_{n_1}(x_1)}{(x_1 - x_{i_1})w_{n_1}^1(x_{i_1})} dx_1 \int_{a_2}^{A_2} \frac{w_{n_2}(x_2)}{(x_2 - x_{i_2})w_{n_2}^1(x_{i_2})} dx_2 ... \int_{a_n}^{A_n} \frac{w_{n_n}(x_n)}{(x_n - x_{i_n})w_{n_n}^1(x_{i_n})} dx_n +$$
$$\iint_{(\Omega)} ...\int R dx_1 dx_2 ... dx_n. \qquad (6)$$

Here R is the remainder of interpolation formula (6).

In the last equation, introducing the notation,

$$C_{i_k}^{(l)} = \int_{a_K}^{A_k} \frac{w_{n_k}(x_k)}{(x_k - x_{i_k})w'_{n_k}(x_{i_k})} dx_k, \qquad k = \overline{1,n}; \ l = \overline{1,n};, \qquad (7)$$

we have

$$J = \iint_{(\Omega)}...\int f(x_1, x_2, ..., x_n)dx_1dx_2,...,dx_n = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}...\sum_{i_n=0}^{n_n} C_{i_1}^{(1)}C_{i_2}^{(2)}...$$
$$...C_{i_n}^{(n)} f(x_{i_1}, x_{i_2}, ..., x_{i_n}) + \iint_{(D)}\int R dx_1 dx_2 ... dx_n. \qquad (8)$$

$C_{i_k}^{(l)}$ are the coefficients of numerical integration formulas of homogeneous integrals.

On this basis in the work [4] and introducing the notation

$$q_k = \frac{x_k - x_0^{(k)}}{h_k},$$
$$q_k^{[n_k+1]} = q_k(q_k-1)...(q_k-n_k), \quad k = 1,2,...,n \tag{9}$$

and coefficients (7) , we reduce to

$$C_i^{(l)} = (A_l - a_l)\cdot H_i^{(l)}, \quad i = 0,1,...,n; \quad l = \overline{1,n}, \tag{10}$$

$$H_i^{(l)} = \frac{1}{n_l}\frac{(-1)^{n_l-i_l}}{i_k!(n_l-i_l)!}\int_0^{n_l}\frac{q^{[n_l+1]}}{q_l-i_l}dq_l,\ i_l = 0,1,2,...,n_l;\ l = \overline{1,n}. \tag{11}$$

Now, let' consider the special cases. Calculating the double integrals' values.

2 Double integrals' values' calculation algorithm's description

Theorem 1. *Suppose that the* $f(x_1, x_2)$ *function is defined and continuous in a given region of integration. Then the cubature formula for the double integral values, obtained by replacing the integrand by a polynomial interpolation for* $n_1 = 1$ *and* $n_2 = 1$ *has the form*

$$J = \iint_{(D)} f(x_1,x_2)dx_1dx_2 =$$
$$\frac{h_x}{2}\frac{h_y}{2}[f(x_0,y_0) + f(x_0,y_1) + f(x_1,y_0) + f(x_1,y_1)] =$$
$$= \frac{h_x h_2}{h}[f_{0,0} + f_{0,1} + f_{10} + f_{11}].$$

Proof. In proving theorems we use equations (8) - (11). In this case, (8) and (11) respectively have the form

$$J = \iint_{(D)} f(x_1,x_2)dx_1dx_2 = \int_{a_1}^{A_1}\int_{a_2}^{A_2} f(x,y)dxdy = \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} C_{i_1}^{(1)}C_{i_2}^{(2)} f(x_{i_1}, x_{i_2}), \tag{13}$$

$$\left.\begin{aligned} H_{i_1}^{(1)} &= \frac{1}{n_1}\frac{(-1)^{n_1-i_1}}{i_1!(n_1-i_1)!}\int_0^{n_1}\frac{q_1^{[n_1+1]}}{q_1-i_1}dq_1,\ i_1=0,1,2,...,n_1;\\ H_{i_2}^{(2)} &= \frac{1}{n_2}\frac{(-1)^{n_2-i_2}}{i_2!(n_2-i_2)!}\int_0^{n_2}\frac{q_1^{[n_1+1]}}{q_2-i_2}dq_2,\ i_2=0,1,2,...,n_2.\end{aligned}\right\}$$

Calculate the values of H_{i_1} and H_{i_2}, for $n_1=1$ and $n_2=1$ respectively:

$$H_0^{(1)}=\frac{1}{1}\cdot\frac{(-1)^{1-0}}{0!(1-0)!}\int_0^1\frac{q_1(q_1-1)}{q_1-0}dq_1=-\int_0^1(q_1-1)dq_1=-\frac{(q_1-1)^2}{2}\bigg|_0^1=$$

$$=-\frac{1}{2}[0-1]=\frac{1}{2}.$$

$$H_1^{(1)}=\frac{1}{1}\cdot\frac{(-1)^{1-1}}{1!0!}\int_0^1\frac{q_1(q_1-1)}{q_1-1}dq_1=\int_0^1 q_1dq_1=\frac{q_1^2}{2}\bigg|_0^1=\frac{1}{2}.$$

In the same way we calculate the values $H_0^{(2)}$ and $H_1^{(2)}$:

$$H_0^{(2)}=\frac{1}{2},\ H_1^{(2)}=\frac{1}{2}.$$

On the basis of $H_0^{(1)},H_1^{(2)}$ we compute $C_0^{(1)}$ and $C_1^{(1)}$, as well as on the $H_0^{(2)}$ and $H_1^{(2)}$ base ,the $C_0^{(2)}$ and $C_1^{(2)}$ value and taking into account (10):

$$C_0^{(1)}=C_1^{(1)}=\frac{h_x}{2};\ C_0^{(2)}=C_1^{(2)}=\frac{h_y}{2}. \tag{15}$$

Here $f_{i,j}=f(x_i,x_j), i,j=\overline{0,1}.$

Substituting this result in (13), we obtain (12).

This result coincides with the result, obtained by repeated application of the trapezoidal quadrature rule, which shows the equivalence of the method of the integrand interpolation replacing by polynomial method and re-use of quadrature formulas.

Theorem 2. *Suppose that the $f(x_1,x_2)$ function is defined and continuous in the closed area of integration. Then the cubature formula for the double integral* *values, obtained by replacing the integrand by a polynomial interpolation for* $n_1 = 2,\ n_2 = 2$ *has the form*

$$J = \iint\limits_{(D)} f(x_1, x_2)dx_1dx_2 = \frac{h_x h_y}{9}[(f_{00} + f_{20} + f_{02} + f_{22}) + 4(f_{1,0} + f_{0,1} + f_{2,1} + f_{1,2}) + 16 f_{11}]. \quad (16)$$

Proof. For this purpose, we calculate the $C_{i_1}^{(1)}$ and $C_{i_2}^{(2)}$ values now for $n_1 = 2$ and $n_2 = 2$:

$$H_0^{(1)} = \frac{1}{2}\frac{(-1)^{2-0}}{0!(2-0)!}\int_0^2 \frac{(q_1-1)}{q_1-0}dq_1 = \frac{1}{2}\frac{1}{2}\int_0^2 (q_1-1)(q_1-2)dq_1 =$$

$$= \frac{1}{4}\int_0^2 (q_1^2 - 3q_1 + 2)dq_1 = \frac{1}{4}\left(\frac{q_1^3}{3} - 3\frac{q_1^2}{2} + 2q_1\right)_0^2 = \frac{1}{4}\left(\frac{8}{3} - 6 + 4\right) = \frac{1}{6},$$

$$H_1^{(1)} = \frac{(-1)^{2-1}}{1!(2-1)!}\int_0^2 \frac{q_1(q_1-1)(q_1-2)}{q_1-1}dq_1 = -\frac{1}{2}\cdot\frac{1}{1}\cdot\int_0^2 q_1\cdot(q_1-2)dq_1 =$$

$$= -\frac{1}{2}\int_0^2 (q_1^2 - 2q_1)dq_1 = -\frac{1}{2}\left(\frac{q_1^3}{3} - q_1^2\right)\Bigg|_0^2 = -\frac{1}{2}\left(\frac{8}{3} - 4\right) = \frac{2}{3},$$

$$H_2^{(1)} = \frac{1}{2}\frac{(-1)^{2-2}}{2!(2-2)!}\int_0^2 \frac{q_1(q_1-1)(q_1-2)}{q_1-2}dq_1 = \frac{1}{2}\cdot\frac{1}{2}\cdot\int_0^2 (q_1^2 - q_1)dq_1 = \frac{1}{4}\left(\frac{q_1^3}{3} - \frac{q_1^2}{2}\right) =$$

$$= \frac{1}{4}\int_0^2\left(\frac{8}{3} - 2\right) = \frac{1}{6}.$$

In the same way, for $H_i^{(2)}\ (i = 0,1,2)$ we have

$$H_0^{(2)} = \frac{1}{6}, H_1^{(2)} = \frac{2}{3}, H_2^{(2)} = \frac{1}{6}.$$

Then ,based on the $H_{i_1}^{(1)}$ and $H_{i_2}^{(2)}$ values,we calculate $C_{i_1}^{(1)}$ and $C_{i_2}^{(2)}$. Then, substituting the $C_{i_1}^{(1)}$ and $C_{i_2}^{(2)}$ values in (13), we obtain (16).

Equation (16) coincides with the result, obtained, using the re-use of Simpson’s quadrature formulas. This fact confirms the equivalence of double integrals, obtained by the repeated application of quadrature formulas and methods of replacing the integrand by interpolating polynomial ,computing.

Let's rewrite (12) and (16) ,respectively, in the form of a component as:

$$J_T = \iint_D f(x_1, x_2)dx_1dx_2 = \frac{h_x h_y}{4}\sum_{i=0}^{1}\sum_{i=0}^{1} f_{ij}\ , \tag{17}$$

$$J_S = \iint_D f(x_1, x_2)dx_1dx_2 = \frac{h_x h_y}{9}\sum_{i_1=0}^{2}\sum_{i_2=0}^{2}(C_2^{i_1})^2(C_2^{i_2})^2 f_{ij}\ . \tag{18}$$

To calculate the n-fold integrals, using (7), we can derive the corresponding formulas.

Theorem 3. *Let the* $f(x_1, x_2,..., x_n)$ *functions be defined and continuous in the closed domain of integration* Ω_n.. *Then n-dimensional cubature formulas for* $f(x_1, x_2,..., x_n)$ *for* $n_1 = n_2 = ... = n_n = 1$ *and* $n_1 = n_2 = ... = n_n = 2$ *approximate calculations will relatively take the form*

$$J = \iint_{(D)}...\int f(x_1, x_2,..., x_n)dx_1dx_2...dx_n = \frac{h_1h_2...h_n}{2^n}\sum_{i_1=0}^{1}\sum_{i_1=0}^{1}...\sum_{i_n=0}^{1} f_{i_1i_2...i_n}\ , \tag{19}$$

where $h_i = A_i - a_i$;

$$J = \iint_{(D)}...\int f(x_1, x_2,..., x_n)dx_1dx_2...dx_n = \frac{h_1h_2...h_n}{3^4}\sum_{i_1=0}^{2}\sum_{i_1=0}^{2}...\sum_{i_n=0}^{2}(C_2^{i_1})^2(C_2^{i_2})^2...(C_2^{(i_2)})^2 f_{i_1i_2...i_n}\ , \tag{20}$$

Here $h_i = \dfrac{A_i - a_i}{2}$.

The proof of this theorem can also be made by induction.

(19) and (20), respectively, coincide with the formulas ,obtained by the repeated application of the quadrature trapezoidal and Simpson, and are given in [2].

Based on these results, let's make the following conclusions.

1. If we put $n_1 = n_2 = ... = n_n = 1$ to (7) , we obtain cubature formulas of the form (19), which coincides with the formula, obtained by repeated application of a quadrature trapeze formula to calculate values of multiple integrals.

2. If we put $n_1 = n_2 = ... = n_n = 2$ to (7), we obtain formulas for the approximate calculation of n-fold integrals of the form (20), which coincides with the formula, obtained by repeated application of the Simpson quadrature formula to compute the values of multiple integrals.

3. If we put $n_1 = n_2 = ... = n_n = k(k > 2)$ to (7), we obtain the Newton-Cotes high order formulas for the approximate calculation of n-fold integrals, and thus provide a high order of accuracy of n-fold cubature formulas.

4.If we put

$$n_1 = S_1, n_2 = S_2, ..., n_n = S_n \text{ to (7)} \tag{21}$$

we ‘ll get the n-fold composite cubature formulas for n-fold integrals.

The essence of (21) means that for each $x_{i_k}\,(k = \overline{1,n})$ we use different Newton-Cotes’ formulas. This is of interest to the construction of n-dimensional cubature formulas.

Theorem 4. *Let $f(x_1, x_2)$ be defined and continuous in the closed area of integration. Then the cubature formula for the approximate calculation of values by replacing the integrand interpolation by polynomial obtained when $n_1 = n_2 = 3$, has the form*

$$\iint_D f(x_1, x_2)dx_1dx_2 = dx = \sum_{i_1=0}^{3} \sum_{i_2=0}^{3} C_{i_1}^{(1)} C_{i_2}^{(2)} f(x_{i_1}, x_{i_2}) = \frac{(h_1)(h_2)}{64}\{[f(x_0, y_0) + f(x_0, y_3) +$$

$$+ f(x_3, y_0) + f(x_3, y_2)] + 3[f(x_0, y_1) + f(x_0, y_2) + f(x_1, y_0) + f(x_1, y_3) + f(x_2, y_0) +$$

$$+ f(x_1, y_3) + f(x_3, y_2) + f(x_3, y_2)] + 9[f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_2) + f(x_2, y_2)]\} = \tag{22}$$

$$= \frac{9}{64}(h_1h_2)\sum_{i_1=0}^{3} \sum_{i_2=0}^{3} C_3^{i_1} C_3^{i_2} f(x_{i_1}, x_{i_2}),$$

where $h_i = \dfrac{A_i - a_i}{3}, i = \overline{1,2}.$

Proof. For this purpose, in accordance with paragraph 3, we derive cubature formulas to calculate values for double integrals for $n_1 = n_2 = 3$ in (12). In this case, the $H_{i_1}^{(1)}$ and $H_{i_2}^{(2)}$ values, according to (13), take the form

$$H_{i_1}^{(1)} = \frac{1}{3}\cdot\frac{(-1)^{3-i_1}}{i_1!(3-i_1)!}\int_0^3 \frac{q_1^{[3+1]}}{q_1 - i_1} d_{q_1} = \frac{1}{3}\frac{(-1)^{3-i_1}}{i_1(3-i_1)!}\int_0^3 \frac{q_1(q_1-1)(q_1-2)(q_1-3)}{q_1 - i_1} dq_1, i_1 = 0,1,2,3$$

$$H_{i_2}^{(2)} = \frac{1}{3}\cdot\frac{(-1)^{3-i_2}}{i_2!(3-i_2)!}\int_0^3 \frac{q_2(q_2-2)(q_2-2)(q_2-3)}{q - i_2} dq_2, i_2 = 0,1,2,3, \tag{23}$$

$$H_0^{(1)} = \frac{1}{3}\cdot\frac{(-1)^{3-0}}{0!(3-0)!}\int_0^3 \frac{q_1(q_1-1)(q_1-2)(q_1-3)}{q_1 - 0} d_{q_1} = -\frac{1}{3}\frac{1}{3!}\int_0^3 (q_1-1)(q_1-2)(q_1-3)dq_1 =$$

$$= \frac{1}{3}\cdot\frac{1}{3!}\left(\frac{q_1^4}{4} - 2\cdot q_1^3 + 11\cdot\frac{q_1^2}{2} - 6q_1\right)\Bigg|_0^3 = \frac{1}{8},$$

$$H_1^{(1)} = \frac{1}{3}\cdot\frac{(-1)^{3-1}}{1!(3-1)!}\int_0^3 \frac{q_1(q_1-1)(q_1-2)(q_1-3)}{q_1 - 0} d_{q_1} = \frac{1}{3}\frac{1}{2!}\int_0^3 (q_1-2)(q_1-3)q_1 dq_1 =$$

$$= -\frac{1}{3}\cdot\frac{1}{2!}\left(\frac{q_1^4}{4} - 5\cdot\frac{q_1^3}{3} + 6\cdot\frac{q_1^2}{2}\right)\Bigg|_0^3 = \frac{3}{8},$$

$$H_2^{(1)} = -\frac{1}{3}\frac{1}{2!1!}\int_0^3 q_1(q_1-1)(q_1-3)dq_1 = -\frac{1}{3}\cdot\frac{1}{2!}\left(\frac{1}{4}q_1^4 + \frac{4}{3}q_1^3 + \frac{3}{2}q_1^2\right)\Bigg|_0^3 = \frac{3}{8}$$

$$H_3^{(1)} = \frac{1}{3}\frac{1}{3!0!}\int_0^3 q_1(q_1-1)(q_1-2)dq_1 = \frac{1}{3}\cdot\frac{1}{3!}\left(\frac{q_1^4}{4} - 3\cdot\frac{q_1^3}{3} + \frac{q_1^2}{2}\right)\Bigg|_0^3 = \frac{1}{8}.$$

In the same way, for $H_i^{(2)}$ $(i = 0,1,2,3)$ we have

$$H_0^{(2)} = \frac{1}{8}, H_1^{(2)} = H_2^{(2)} = \frac{3}{8}, H_3^{(2)} = \frac{1}{8}.$$

Let's rewrite the formula (23) in the form of:

$$\iint\limits_{(D)} f(x_1,x_2)dx_1dx_2 = \int_{a_1}^{A_1}\int_{a_2}^{A_2} f(x_1,x_2)dx_1dx_2 = \frac{9}{64}h_1h_2\sum_{i_1=0}^{3}\sum_{i_2=0}^{3} C_3^{i_1}C_3^{i_2} f_{i_1i_2} \tag{24}$$

where $C_3^{i_1} = \dfrac{3!}{i_1!(3-i_1)!}$, $C_3^{i_2} = \dfrac{3!}{i_2!(3-i_2)!}$.

Theorem 5. *Let* $f(x_1, x_2, ..., x_n)$ *be defined and continuous in n-dimensional integration region* Ω_n. *Then cubature formula for the approximate computation of the values of the* $f(x_1, x_2, ..., x_n)$ *integrand by interpolation polynomial , obtained for* $n_1 = n_2 = ... = n_n = 3$,*, has the form:*

$$\iint\limits_{(D)} \int f(x_1, x_2, ..., x_n) dx_1 dx_2, ..., dx_n = \int\limits_{a_1}^{A_1} \int\limits_{a_2}^{A_2} \int\limits_{a_n}^{A_n} f(x_1, x_2, ..., x_n) dx_1 dx_2, ..., dx_n =$$

$$= \left(\frac{3}{8}\right)^n \prod_{i=1}^{n} h_i \sum_{i=0}^{3} \sum_{i_2=0}^{3} ... \sum_{i_1=0}^{3} C_3^{i_1} C_3^{i_2} ... C_3^{i_n} f_{i_1 i_2, ... i_n}, \tag{25}$$

where $C_3^{i_k} = \dfrac{3!}{i_k!(3-i_k)!}$, $k = \overline{1, n}$.

Proof. In the equations to calculate the values of double integrals (17) and (18), respectively, and produced by $n_1 = n_2 = 1$ and $n_1 = n_2 = 2$, and the values of n-fold integrals (19) and (20), respectively, and produced by $n_1 = n_2 = ... = n_n = 1$ и $n_1 = n_2 = ... = n_n = 2$, and (24) to calculate the values of n-fold integrals as $n_1 = n_2 = ... = n_n = 3$, are useful when the size of the region is very small. If the $A_i - a_i$ dimensions are large, then to increase the accuracy of the n-fold cubature formulas of each $[a_i, A_i]$, we ‘ll divide the system under the

$$[a_i, A_i] = [a_i = a_i^{(0)}, a_i^{(1)}] \cup [a_i^{(1)} \cup a_i^{(2)}] \cup ... \cup [a_i^{(k-1)}, a_i^k = A_i]$$

integrals, and apply to each the appropriate cubature formulas.

To understand this approach fully, we’ll first consider the double integrals. Here the two $[a_1, A_1]$ and $[a_2, A_2]$ interpolations correspond to directions along the ox_1 and ox_2. axes The $[a_1, A_1]$ intervals are divided by n_1, and the $[a_2, A_2]$ interval – by n_2 parts. Next , for $n_1 = n_2 = 2$, we have

$$h_2 = h_1 = \frac{A_1 - a_1}{n_1}, \quad h_1 = h_2 = \frac{A_2 - b_2}{n_2} \tag{26}$$

And for $n_1 = n_2 = 2$

$$h_2 = h_1 = \frac{h - a_1}{2n_1}, h_y = h_2 = \frac{A_2 - b_2}{2n_2}. \qquad (27)$$

Taking into account (25) and (26), the (17) and (18) can be respectively rewritten as

$$\iint_D f(x_1, x_2) dx_1 dx_2 = \int_{a_1}^{A_1} \int_{a_2}^{A_2} f(x_1, x_2)\, dx_1 dx_2 = \frac{h_1 h_2}{4} \sum_{i_i=0}^{n_1} \sum_{i_2=0}^{n_2} \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} f_{i_1+i_1, i_2+i_2}, \qquad (28)$$

$$\iint_D f(x_1, x_2) dx_1 dx_2 = \int_{a_1}^{A_1} \int_{\alpha_2}^{A_2} f(x_1, x_2)\, dx_1 dx_2 = \frac{h_1 h_2}{9} \sum_{i_i=0}^{n_1} \sum_{i_2=0}^{n_2} \sum_{j_1=0}^{2} \sum_{j_2=0}^{2} (C_2^{i_1})^2 (C_2^{i_2})^2 f_{i_1+i_1, i_2+i_2}. \qquad (29)$$

Reasoning by analogy (28) and (29), the formulas (19), (20) and (24) respectively, are rewritten as following:

$$\iint_D \ldots \int f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n = \int_{a_1}^{A_1} \int_{a_2}^{A_2} \ldots \int_{a_n}^{A_n} f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n =$$
$$= \prod_{i=1}^{n} \left(\frac{h_i}{2}\right) \sum_{i_i=0}^{h_1} \sum_{i_2=0}^{h_2} \ldots \sum_{i_n=0}^{h_n} \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} \ldots \sum_{j_n=0}^{1} f_{i_1+j_1, i_2+j_2, \ldots, i_n+j_n}, \qquad (30)$$

$$\iint_D \ldots \int f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n = \int_{a_1}^{A_1} \int_{a_2}^{A_2} \ldots \int_{a_n}^{A_n} f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n = \qquad (31)$$
$$= \prod_{i=1}^{n} \left(\frac{a_i}{3}\right) \sum_{i_i=0}^{2n_1} \sum_{i_2=0}^{2n_2} \ldots \sum_{i_n=0}^{2n_n} \sum_{j_1=0}^{2} \sum_{j_2=0}^{2} \ldots \sum_{j_n=0}^{2} (C_2^{i_1})^2 (C_2^{i_2})^2 \ldots (C_2^{i_n})^2 f_{i_1+j_1, i_2+j_2, \ldots, i_n+j_n},$$

$$\iint_{(D)} \ldots \int f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n = \int_{a_1}^{A_1} \int_{a_2}^{A_2} \ldots \int_{a_n}^{A_n} f(x_1, x_2, \ldots, x_n)\, dx_1 dx_2 \ldots dx_n =$$
$$= \prod_{i=1}^{h} h_i \sum_{i_i=0}^{n_1} \sum_{i_2=0}^{n_2} \ldots \sum_{i_n=0}^{n_n} \sum_{j_1=0}^{3} \sum_{j_2=0}^{3} \ldots \sum_{j_n=0}^{3} C_3^{i_1} C_3^{i_2} \ldots C_3^{i_n} f_{i_1+j_1, i_2+j_2, \ldots, i_n+j_n}.$$

It should be noted that in the calculation of multiple $n_1 = n_2 = \ldots = n_n = 2$ integrals, $C_2^{i_k}$ will be present under the sum, and when $n_1 = n_2 = \ldots = n_n = 3$ - $C_3^{i_k}$ etc. Note that this rule, since $n_1 = n_2 = \ldots = n_n = 4$ is violated. For clarity, we present below the Cotes coefficients (Table 1).

According to the values of numbers, standing in rows and columns (see Table 1), the Cotes coefficients are increasing rapidly with the increasing of values and, since $n = 8$,become negative. Therefore, we have developed a program in Maple 11 mathematical system for the coefficients' and constructed square Newton - Cotes formulas' calculating. We now give a quadrature formulas, constructed by using this Maple procedure.

In Table 1 (for convenience), according to [4], the Coates recording rate for each $\hat{H}_1$ is presented in the form of fractions: $H_i = \frac{H_i}{N}$ with a common factor N. It is clear ,that $\sum_{i=n}^{n} \widehat{H}_i = N.$ According Table 1, with large values,the results may be negative.

Now, the value of this integral is calculated approximately by the Newton-Cotes, when n = 6 (seven ordinates)

Example. Let's calculate

$$Y = \int_0^1 \int_0^1 y \cdot \frac{1}{x+1} dxdy .$$

Table 1

n	$\hat{H}_1$	$\hat{H}_2$	$\hat{H}_3$	$\hat{H}_4$	$\hat{H}_5$	$\hat{H}_6$
1	1					
2	4	1				
3	3	3	1			
4	32	12	32	7		
5	75	50	50	75	19	
6	216	27	272	27	216	41
7	3577	1323	2989	2989	1313	3572
8	5888	-928	10469	-4540	10496	-928

9	15741	1080	19344	5778	5778	19344
10	106300	-48525	272400	260550	472368	-260550
11	13486539	-3237113	25226685	-9595542	15493566	15493566
12	9903168	-7587864	35725120	-5149130	87516288	87797136

Continuation of table 1

$\hat{H}_7$	$\hat{H}_8$	$\hat{H}_9$	$\hat{H}_{10}$	$\hat{H}_{11}$	$\hat{H}_{12}$	$\hat{H}_{13}$	$\hat{H}_{14}$	$\hat{H}_{15}$	Value objects
									2
									6
									8
									90
									288
									840
751									17280
5888	989								28350
1080	15741	2857							89600
272400	106300	106300	16067						598732
-9595542	25226685	-3237113	13486539	2171465					87091200
87516288	-5149130	35725120	-7587864	9903168	1364651				63063000

The exact value of the integral is the following:

$$I_T = \int_0^1 \frac{dx}{x+1}\int_0^1 ydy = \ell n|x+1|\Big|_0^1 \frac{y^2}{2}\Big|_0^1 = \ln 2\cdot\frac{1}{2} = \frac{\ln 2}{2} = \frac{0{,}7935}{2} = 0{,}3467,$$

$$h_1 = h_x = \frac{1-0}{6} = \frac{1}{6};\; h_2 = h_y = \frac{1-0}{6} = \frac{1}{6}$$

Let’s draw up a table of values (Table 2), assuming that $H_i = 840A_i$.

Table 2:. Calculating values.

i	x_i	$f_1(x_i)=\frac{1}{1+x_i}$	y	$f_2=y_i$	H_i	$H_i f_1(x_i)$	$H_i f_2(y_i)$
0	0	0	0	0	41	41	0
1	$\frac{1}{6}$	$\frac{6}{7}$	$\frac{1}{6}$	$\frac{1}{6}$	216	185,14257	$\frac{26}{72}=36$
2	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	27	20,25	$\frac{27}{18}=\frac{3}{2}=9$
3	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	272	181,3333	$\frac{272}{4}=136$
4	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{1}{3}$	27	16,2	$\frac{327\cdot y^2}{189}=1$
5	$\frac{5}{6}$	$\frac{6}{11}$	$\frac{5}{6}$	$\frac{1}{6}$	216	117,81918 2	$\frac{25\cdot 26}{72}=180$
6	7	$\frac{1}{2}$	1	2	41	20,25	$\frac{41}{2}=41$
Σ						581,99437 2	420

Now, based on this table we calculate the value of double integral:

$$I_{np} = \frac{1}{840}\cdot 591{,}994372\cdot\frac{420}{940} = 0{,}6933\cdot\frac{1}{2} = 0{,}3466$$

despite the fact that the values I_T (precision) and I_{np} (approximate) are close enough. Along with this, we note that the n of Cotes coefficient are negative, large (in absolute value) that are very different from each other. Therefore, the Newton - Cotes is not recommended for large n.

Instead, when n = 1 or n = 2 or n = 3 each integration interval should be divided into n parts and use the appropriate Newton-Cotes in each of them.

3 Conclusions

Thus we have constructed the multi-dimensional cubature formulas to approximate the values of multiple integrals by replacing the integrands by interpolation polynomial, which can be easily implemented on a computer.

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