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$M^{[X]}/G/1$ with Bernoulli Schedule Server Vacation Random Break Down and second optional Repair

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Abstract

We present a single server subject to random breakdowns followed by a repair and Bernoulli scheduled server vacation. The customers arrive in batches and whose service being provided one by one according to first come first served discipline. Upon completion of a service, the server will go for vacation with probability p or remain staying back in the system for providing the service to the next customer with probability 1-p, if any. Both service time and vacation time follow general (arbitrary) distribution. The system may experience breakdown at random time and the breakdowns occur according to Poisson stream. Once the server breakdown, it must be send to repair process immediately. The most realistic aspect in modeling of a unreliable server, multi optional repair may be required. If the server could not be repaired or restored with the first essential repair, subsequent repairs are needed for the restoration of the server. Both essential and optional repair times follow exponential distribution. We obtain the time dependent probability generating functions in terms of their Laplace transforms

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and the corresponding steady state results explicitly. Also we derive the average number of customers in the queue and the average waiting time in closed form.

Mathematics Subject Classification: 60K25; 60K30
Keywords: Bernoulli vacation; probability generating function; first essential repair; second optional repair; steady state; mean queue length; mean waiting time

1 Introduction

Queueing modeling is being used enormously and effectively in congestion problems which are encountered in real life such as waiting lines at airports, railway stations, banks as well as industrial situations which includes computer systems, web services and communication networks etc. There is a vast literature in the research of queueing theory, because of its wide applicability in modeling over congestion problems. The queueing model with server vacations (server absences) has been well studied and successfully applied in many areas such as production, servicing, computer and communication network systems. A remarkable and excellent surveys on the earlier works of vacation models have been reported by Doshi (1986), Takagi (1991). Many authors including Choudhury and Madhan (2004), Anabosi et al.(2003) incorporated the concept of Bernoulli scheduled server vacation on non-Markovian queues.

In queueing theory parlance, temporary periods of unavailability of service are referred to as server vacations, server interruptions or server breakdowns. Server break down is a great issue as it makes negative impact on the system performance. So it is important to have a reliable server in order to maintain the quality of service. Maraghi et al.(2009) have obtained steady state solution of batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general vacation time. The most realistic aspect in modeling of a unreliable server, multi optional repair which have been discussed by Madhu jain et al.(2011). When server could not be repaired or restored by the first essential repair, subsequent repairs are needed to restore the server. Hsieh, Yi-Chih and Andersl(1995) studied a queueing model in which the server is subject to several types of breakdowns, and each type has two possible stages of repair. Gray, Wang and Scott(2004) studied a queueing Model with multiple types of server breakdowns in which each type of breakdown requires a finite random number of stages of repair.

In this paper we consider a queueing system wherein the customers arrive in batches and the server provides service one by one in FCFS basis. As soon as the service of a customer is completed, the server may go for a vacation with probability p or continue staying in the system to provide service to a next customer, if any, with probability $1-p$. On account of, the system may subject to breakdowns during busy time, the breakdowns occur according to Poisson process with mean break down rate $\alpha(>0)$. Once the system breakdown, it is immediately sent for repair wherein the repairman or repairing apparatus provides the first essential repair (FER). After the completion of FER, the server may opt for second optional repair (SOR) with probability r or may join the system with complementary probability 1-r to render the service to the customers. After the completion of the required repair, the server provides service with the same efficiency as before failure according to FCFS discipline. Both first essential and second optional repair follow exponential with mean $\frac{1}{\beta_1}$ and $\frac{1}{\beta_2}$ respectively. After the repair process complete, the server resumes its work immediately. Also whenever the system meet a break down, the customer whose service is interrupted goes back to the head of the queue.

The rest of the paper is organized as follows. The mathematical description of our model is in Section 2 and equations governing the model are given in Section 3. The time dependent solution have been obtained in Section 4, the corresponding steady state results have been derived explicitly in Section 5 and the concluding remarks is in 6.

2 Mathematical Description of the model

We assume the following to describe the queueing model of our study.

• Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i \triangle t$ (i=1,2,3,....) be the first order probability that a batch of i customers arrives at the system during a short interval of time

$(t, t + \triangle t)$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. The customers are served one-by-one on a "first come-first served" basis.

• Each customer undergoes service provided by a single server on a first come first served basis. The service time follows different general (arbitrary) distributions with distribution function $B(v)$ and the density function $b(v)$.

• Let $\mu(x)$dx be the conditional probability of completion of the service during the interval (x, x+ dx] given that elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)} \tag{1}$$

and therefore,

$$b(v) = \mu(v) e^{-\int_0^v \mu(x) dx} \tag{2}$$

•As soon as, service of a customer is completed, the server may go for a vacation of random length V with probability p $(0 \leq p \leq 1)$ or it may continue to serve the next customer $(1 - p)$.

•The vacation time also follow general (arbitrary) distribution with distribution function V(s) and the density function v(s). Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval (x, x + dx] given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1 - V(x)} \tag{3}$$

and therefore,

$$v(s) = \gamma(s) e^{-\int_0^s \gamma(x) dx} \tag{4}$$

• On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.

• The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$.

•As soon as the server is broken down, it is immediately sent for repair wherein the repairman or repairing apparatus provides the first essential repair (FER). After the completion of FER, the server may opt for second optional repair (SOR) with probability r or may join the system with complementary probability 1-r to render the service to the customers.

• The repair process provides two types of repair in which the first type of repair is essential and the second type of repair is optional. Both exponentially distributed with mean $\frac{1}{\beta_1}$. and $\frac{1}{\beta_2}$. After the completion of the required repair, the server provides service with the same efficiency as before failure according to FCFS discipline.

• Various stochastic processes involved in the system are assumed to be independent of each other.

3 Definitions and equations governing the system

We let,

(i) $P_n(x,t)$ = Probability that at time 't' the server is active providing service and there are 'n' $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in the service irrespective of the value of x.

(ii) $V_n(x,t)$ = probability that at time 't' the server is on vacation with elapsed vacation time x, and there are 'n' $(n \geq 0)$ customers waiting in the queue for service. Consequently $V_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

(iii) $R_n^{(1)}(t)$ = Probability that at time t, the server is inactive due to breakdown and the system is under first essential repair while there are 'n' $(n \geq 0)$ customers in the queue.

(iv) $R_n^{(2)}(t)$ = Probability that at time t, the server is inactive due to breakdown and the system is under second optional repair while there are 'n' $(n \geq 0)$ customers in the queue.

(v) $Q(t)$ = probability that at time 't' there are no customers in the system and the server is idle but available in the system.

The queueing model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial t}P_n(x,t) + \frac{\partial}{\partial x}P_n(x,t) + (\lambda + \mu(x) + \alpha)P_n(x,t) = \lambda \sum_{i=1}^{n-1} c_i P_{n-i}(x,t), \quad n \geq 1 \tag{5}$$

$$\frac{\partial}{\partial t}P_0(x,t) + \frac{\partial}{\partial x}P_0(x,t) + (\lambda + \mu(x) + \alpha)P_0(x,t) = 0 \tag{6}$$

$$\frac{\partial}{\partial t}V_n(x,t) + \frac{\partial}{\partial x}V_n(x,t) + (\lambda + \gamma(x))V_n(x,t) = \lambda \sum_{i=1}^{n-1} c_i V_{n-i}(x,t) \quad n \geq 1 \tag{7}$$

$$\frac{\partial}{\partial t}V_0(x,t) + \frac{\partial}{\partial x}V_0(x,t) + (\lambda + \gamma(x))V_0(x,t) = 0 \tag{8}$$

$$\frac{d}{dt}R_n^{(1)}(t) = -(\lambda + \beta_1)R_n^{(1)}(t) + \lambda \sum_{i=1}^{n} c_i R_{n-i}^{(1)}(t) + \alpha \int_0^\infty P_{n-1}(x,t)dx \tag{9}$$

$$\frac{d}{dt}R_0^{(1)}(t) = -(\lambda + \beta_1)R_0^{(1)}(t) \tag{10}$$

$$\frac{d}{dt}R_n^{(2)}(t) = -(\lambda + \beta_2)R_n^{(2)}(t) + \lambda \sum_{i=1}^{n} c_i R_{n-i}^{(2)}(t) + r\beta_1 R_n^{(1)}(t) \tag{11}$$

$$\frac{d}{dt}R_0^{(2)}(t) = -(\lambda + \beta_2)R_0^{(2)}(t) + r\beta_1 R_0^{(1)}(t) \tag{12}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t) + (1-r)\beta_1 R_0^{(1)}(t) + \beta_2 R_0^{(2)}(t) + (1-p)\int_0^\infty P_0(x,t)\mu(x)dx + \int_0^\infty V_0(x,t)\gamma(x)dx \tag{13}$$

Equations (5) to (13) are to be solved subject to the following boundary conditions.

$$P_0(0,t) = c_1\lambda Q(t) + (1-r)\beta_1 R_1^{(1)}(t) + \beta_2 R_1^{(2)}(t) + \int_0^\infty V_1(x,t)\gamma(x)dx + (1-p)\int_0^\infty P_1(x,t)\mu(x)dx \quad (14)$$

$$P_n(0,t) = c_{n+1}\lambda Q(t) + (1-r)\beta_1 R_{n+1}^{(1)}(t) + \beta_2 R_{n+1}^{(2)}(t) + \int_0^\infty V_{n+1}(x,t)\gamma(x)dx + (1-p)\int_0^\infty P_{n+1}(x,t)\mu(x)dx \quad (15)$$

$$V_n(0,t) = p\int_0^\infty P_n(x,t)\mu(x)dx, \qquad n \geq 0 \quad (16)$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n(0) = 0; \;\; n = 0,1,2...,; \;\; V_0(0) = V_n(0) = 0; \;\; Q(0) = 1 \quad (17)$$

4 Probability Generating functions of the queue length: The time-dependent solution

We define the probability generating function

$$P_q(x,z,t) = \sum_{n=0}^{\infty} z^n P_n(x,t); P_q(z,t) = \sum_{n=0}^{\infty} z^n P_n(t) \quad (18)$$

$$V_q(x,z,t) = \sum_{n=0}^{\infty} z^n V_n(x,t); V_q(z,t) = \sum_{n=0}^{\infty} z^n V_n(t) \quad (19)$$

$$R_q^{(1)}(z,t) = \sum_{n=0}^{\infty} z^n R_n^{(1)}(t); R_q^{(2)}(z,t) = \sum_{n=0}^{\infty} z^n R_n^{(2)}(t) \quad (20)$$

$$C(z) = \sum_{n=1}^{\infty} c_n z^n, \quad (21)$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function f(t) as

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt \quad (22)$$

Taking Laplace transforms of equations (5) to (13)

$$\frac{\partial}{\partial x}\bar{P}_n(x,s) + (s+\lambda+\mu(x)+\alpha)\bar{P}_n(x,s) = \lambda\sum_{i=1}^{n-1} c_i\bar{P}^{(1)}_{n-i}(x,s) \quad n\geq 1 \tag{23}$$

$$\frac{\partial}{\partial x}\bar{P}_0(x,s) + (s+\lambda+\mu(x)+\alpha)\bar{P}_0(x,s) = 0 \tag{24}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + (s+\lambda+\gamma(x))\bar{V}_n(x,s) = \lambda\sum_{i=1}^{n-1} c_i\bar{V}_{n-i}(x,s) \tag{25}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + (s+\lambda+\gamma(x))\bar{V}_0(x,s) = 0 \tag{26}$$

$$(s+\lambda+\beta_1)\bar{R}^{(1)}_n(s) = \lambda\sum_{i=1}^{n-1} c_i\bar{R}^{(1)}_{n-i}(s) + \alpha\int_0^\infty \bar{P}_{n-1}(x,s)dx \tag{27}$$

$$(s+\lambda+\beta_1)\bar{R}^{(1)}_0(s) = 0 \tag{28}$$

$$(s+\lambda+\beta_2)\bar{R}^{(2)}_n(s) = \lambda\sum_{i=1}^{n-1} c_i\bar{R}^{(2)}_{n-i}(s) + r\beta_1\bar{R}^{(1)}_n(s) \tag{29}$$

$$(s+\lambda+\beta_2)\bar{R}^{(2)}_0(s) = r\beta_1\bar{R}^{(1)}_0(s) \tag{30}$$

$$\begin{aligned}(s+\lambda)\bar{Q}(s) = 1 + (1-r)\beta_1\bar{R}^{(1)}_0(s) + \beta_2\bar{R}^{(2)}_0(s) + \\ + \int_0^\infty \bar{V}_0(x,s)\gamma(x)dx + (1-p)\int_0^\infty \bar{P}_0(x,s)\mu(x)dx\end{aligned} \tag{31}$$

for the boundary conditions

$$\begin{aligned}\bar{P}_0(0,s) = (1-p)\int_0^\infty \bar{P}_1(x,s)\mu(x)dx + \int_0^\infty \bar{V}_1(x,s)\gamma(x)dx \\ + (1-r)\beta_1\bar{R}^{(1)}_1(s) + \beta_2\bar{R}^{(2)}_1(s) + \lambda c_1\bar{Q}(s)\end{aligned} \tag{32}$$

$$\begin{aligned}\bar{P}^{(1)}_n(0,s) = (1-p)\int_0^\infty \bar{P}_{n+1}(x,s)\mu(x)dx + \int_0^\infty \bar{V}_{n+1}(x,s)\gamma(x)dx \\ + (1-r)\beta_1\bar{R}^{(1)}_{n+1}(s) + \beta_2\bar{R}^{(2)}_{n+1}(s) + \lambda c_{n+1}\bar{Q}(s)\end{aligned} \tag{33}$$

$$\bar{V}_n(0,s) = p\int_0^\infty \bar{P}_n(x,s)\mu_2(x)dx; \quad n = 0,1,2,... \tag{34}$$

Now multiplying equation (23) by z^n and summing over n from 1 to ∞, adding to equation (24) and using the definition of probability generating function defined in equation (18), we obtain

$$\frac{\partial}{\partial x}\bar{P}_q(x,z,s) + (s+\lambda-\lambda C(z)+\mu(x)+\alpha)\bar{P}_q(x,z,s) = 0. \tag{35}$$

Performing similar operations on equations (25) to (30)

$$\frac{\partial}{\partial x}\bar{V}_q(x,z,s) + (s+\lambda-\lambda C(z)+\gamma(x))\bar{V}_q(x,z,s) = 0 \tag{36}$$

$$(s+\lambda-\lambda C(z)+\beta_1)\bar{R}_q^{(1)}(z,s) = \alpha z\int_0^\infty \bar{P}_q(x,z,s)dx \tag{37}$$

$$(s+\lambda-\lambda C(z)+\beta_2)\bar{R}_q^{(2)}(z,s) = r\beta_1\bar{R}_q^{(1)}(z,s). \tag{38}$$

For the boundary conditions, multiply both sides of equation (32) by z, multiply both sides of equation (33) by z^{n+1}, summing over 1 to ∞, adding the two results and using the definition of probability generating function, we get,

$$\begin{aligned} z\bar{P}_q(0,z,s) &= (1-p)\int_0^\infty \bar{P}_q(x,z,s)\mu(x)dx + \int_0^\infty \bar{V}_q(x,z,s)\gamma(x)dx \\ &+ (1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s) + (1-r)\beta_1\bar{R}_q^{(1)}(z,s) \\ &+ \beta_2\bar{R}_q^{(2)}(z,s) \end{aligned} \tag{39}$$

Performing similar operation on equation (34) we obtain

$$\bar{V}_q(0,z,s) = p\int_0^\infty \bar{P}_q(x,z,s)\mu(x)dx. \tag{40}$$

Integrating the equation (35) from 0 to x yields

$$\bar{P}_q(x,z,s) = \bar{P}_q(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu(t)dt}, \tag{41}$$

where $\bar{P}_q(0,z,s)$ is given by equation (39).

Again integrating equation (41) by parts with respect to x yields

$$\bar{P}_q(z,s) = \bar{P}_q(0,z,s)\left[\frac{1-\bar{B}(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \tag{42}$$

where

$$\bar{B}(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB(x) \tag{43}$$

is Laplace - Stieltjes transform of the service time $B(x)$. Now multiplying both sides of equation (41) by $\mu(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q(x,z,s)\mu(x)dx = \bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda C(z)+\alpha) \tag{44}$$

Similarly, on integrating equation (36) from 0 to x, we get

$$\bar{V}_q(x,z,s) = p\bar{V}_q(0,z,s)e^{-(s+\lambda-\lambda C(z)x-\int_0^x \gamma(t)dt}, \tag{45}$$

where $\bar{V}_q(0,z,s)$ are given by equations (40).

Again integrating equations (45) by parts with respect to x yields

$$\bar{V}_q(z,s) = p\bar{V}_q(0,z,s)\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z))}\right] \tag{46}$$

where

$$\bar{V}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)x}dV(x) \tag{47}$$

is Laplace - Stieltjes transform of the vacation time $V(x)$.

Now multiplying both sides of equation (45) by $\gamma(x)$ and integrating over x,we get

$$\int_0^\infty \bar{V}q(x,z,s)\gamma(x)dx = p\bar{V}_q(0,z,s)\bar{V}(s+\lambda-\lambda C(z)) \tag{48}$$

Now using equations (44), equation(40) can be written as

$$\bar{V}_q(0,z,s) = p\bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda C(z)+\alpha) \tag{49}$$

Using above equation (49), equation (46) becomes

$$\bar{V}_q(z,s) = p\bar{P}_q(0,z,s)\bar{B}(s+\lambda-\lambda C(z))\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z))}\right] \tag{50}$$

Using equations (44) equation (37) becomes

$$\bar{R}_q^{(1)}(z,s) = \alpha z\bar{P}_q(0,z,s)\left[\frac{1-\bar{B}(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)(s+\lambda-\lambda C(z)+\beta_1)}\right] \tag{51}$$

Using equation (51), equation (38) becomes

$$\bar{R}_q^{(2)}(z,s) = \left[\frac{r\beta_1\alpha z\bar{P}_q(0,z,s)\left(1-\bar{B}(s+\lambda-\lambda C(z)+\alpha)\right)}{(s+\lambda-\lambda C(z)+\alpha)(s+\lambda-\lambda C(z)+\beta_1)(s+\lambda-\lambda C(z)+\beta_2)}\right] \tag{52}$$

Now using equations (44), (48), (51) and (52) in equation (39) and solving for $\bar{P}_q(0,z,s)$ we get

$$\bar{P}_q(0,z,s) = \frac{f_1(z)f_2(z)f_3(z)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{Dr} \tag{53}$$

where

$$\begin{aligned} Dr = {} & f_1(z)f_2(z)f_3(z)\left\{z-[(1-p)+p\bar{V}(s+\lambda-\lambda C(z))]\bar{B}[f_1(z)]\right\} \\ & -\alpha z[(1-r)\beta_1 f_3(z)+r\beta_1\beta_2][1-\bar{B}[f_1(z)]] \end{aligned} \tag{54}$$

$$f_1(z) = s + \lambda - \lambda C(z) + \alpha$$

$$f_2(z) = s + \lambda - \lambda C(z) + \beta_1$$

$$f_3(z) = s + \lambda - \lambda C(z) + \beta_2$$

Substituting the value of $\bar{P}_q(0, z, s)$ from equation (53) in to equation (42), (50), (51) and (52) we get

$$\bar{P}_q(z, s) = \frac{f_2(z)f_3(z)[(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)][1 - \bar{B}[f_1(z)]]}{Dr} \quad (55)$$

$$\bar{V}_q(z, s) = \quad (56)$$
$$\frac{pf_1(z)f_2(z)f_3(z)\bar{B}[f_1(z)][(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)] \left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z))}\right]}{Dr}$$

$$\bar{R}_q^{(1)}(z, s) = f_3(z)\frac{\alpha z[1 - \bar{B}[f_1(z)]][(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr} \quad (57)$$

$$\bar{R}_q^{(2)}(z, s) = r\beta_1\alpha z\frac{[1 - \bar{B}[(z)]][(1 - s\bar{Q}(s)) + \lambda(C(z) - 1)\bar{Q}(s)]}{Dr}, \quad (58)$$

where Dr is given by equation (54).

5 The steady state analysis

In this section we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, suppress the argument't' where ever it appears in the time dependent analysis.

By using well known Tauberian property

$$Lt_{s\to 0}s\bar{f}(s) = Lt_{t\to\infty}f(t), \quad (59)$$

multiplying both sides of equation (55), (56), (57) and (58) by s and applying property (59) then simplifying, we get

$$P_q(z) = \frac{f_2(z)f_3(z)[\lambda(C(z) - 1)][1 - \bar{B}[f_1(z)]]Q}{Dr} \quad (60)$$

$$V_q(z) = p\frac{[f_1(z)f_2(z)f_3(z)\bar{B}[f_1(z)]][\bar{V}(\lambda - \lambda C(z)) - 1]Q}{Dr} \tag{61}$$

$$R_q^{(1)}(z) = \lambda\alpha z\frac{f_3(z)(C(z)-1)[1-\bar{B}[f_1(z)]]Q}{Dr} \tag{62}$$

$$R_q^{(2)}(z) = \frac{r\lambda\alpha\beta_1 z(C(z)-1)[1-\bar{B}[f_1(z)]]Q}{Dr} \tag{63}$$

Let $W_q(z)$ denotes the probability generating function of queue size irrespective of the state of the system. Then adding (60), (61), (62) and (63) we get

$$W_q(z) = P_q(z) + +V_q(z) + R_q^{(1)}(z) + R_q^{(2)}(z) \tag{64}$$

$$\begin{aligned} W_q(z) = &\frac{f_2(z)f_3(z)[\lambda(C(z)-1)][1-\bar{B}[f_1(z)]]Q}{Dr} \\ &+ p\frac{[f_1(z)f_2(z)f_3(z)\bar{B}[f_1(z)]][\bar{V}(\lambda - \lambda C(z)) - 1]Q}{Dr} \\ &+ \lambda\alpha z\frac{f_3(z)(C(z)-1)[1-\bar{B}[f_1(z)]]Q}{Dr} \\ &+ \frac{\lambda\alpha\beta_1(C(z)-1)[1-\bar{B}[f_1(z)]]Q}{Dr} \end{aligned} \tag{65}$$

In order to obtain Q, we use the normalization condition

$$W_q(1) + Q = 1 \tag{66}$$

We see that at $z = 1$, $W_q(z)$ is indeterminate of the form 0/0. We apply L'Hospital's rule in equation (65) where $\bar{B}(0) = 1; \bar{V}(0) = 1, -V'(0) = E[V]$ the mean vacation time.

Now

$$P_q(1) = \frac{\lambda\beta_1\beta_2 Q[1-\bar{B}(\alpha)]E(I)}{dr} \tag{67}$$

$$V_q(1) = p\frac{\lambda\alpha\beta_1\beta_2 Q\bar{B}(\alpha)E(I)E(V)}{dr} \tag{68}$$

$$R_q^{(1)}(1) = \frac{\lambda\alpha\beta_2 QE(I)(1-\bar{B}(\alpha)))}{dr} \tag{69}$$

$$R_q^{(2)}(1) = r\frac{\lambda\alpha\beta_1 QE(I)(1-\bar{B}(\alpha))}{dr} \tag{70}$$

$$W_q(1) = \frac{\lambda QE(I)\left\{(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha)[1 - \bar{B}(\alpha)] + \alpha\beta_1\beta_2 p\bar{B}(\alpha)E(V)\right\}}{dr} \tag{71}$$

where

$$\begin{aligned} dr = {} & \alpha\beta_1\beta_2\bar{B}(\alpha) - \lambda E(I)[(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha)[1 - \bar{B}(\alpha)]] \\ & - p\alpha\beta_1\beta_2 E(I)E(V)\bar{B}(\alpha) \end{aligned} \tag{72}$$

$$Q = 1 - \lambda E(I)\left[\frac{r}{\beta_2\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} + \frac{1}{\beta_1\bar{B}(\alpha)} - \frac{r}{\beta_2} - \frac{1}{\alpha} - \frac{1}{\beta_1} + pE(V)\right] \tag{73}$$

and the the utilization factor ρ of the system is given by,

$$\rho = \lambda E(I)\left[\frac{r}{\beta_2\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} + \frac{1}{\beta_1\bar{B}(\alpha)} - \frac{r}{\beta_2} - \frac{1}{\alpha} - \frac{1}{\beta_1} + pE(V)\right] \tag{74}$$

where $\rho < 1$ is the stability condition under which the steady state exists, equation (73) gives the probability that the server is idle. Substitute Q from equation (73) in equation (65), $W_q(z)$ have been completely and explicitly determined which is the the probability generating function of the queue size.

5.1 The average queue size

Let L_q denote the mean number of customers in the queue under the steady state, then

$$L_q = \frac{d}{dz}W_q(z)\mid_{z=1}$$

since this formula gives 0/0 form, we write

$$W_q(z) = \frac{N(z)}{D(z)},$$

where N(z) and D(z) are the numerator and denominator of the right hand side of equation (65) respectively, then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \tag{75}$$

where primes and double primes in equation (75) denote first and second derivation at $z = 1$ respectively. Carrying out the derivatives at $z = 1$, we have

$$\begin{aligned} N'(1) = {} & \lambda E(I) Q \{ (r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) \\ & + \bar{B}(\alpha)[p\alpha\beta_1\beta_2 E(V) - (r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha)] \} \end{aligned} \tag{76}$$

$$\begin{aligned} N''(1) = {} & 2Q[\lambda E(I)]^2 \left\{ (\frac{\alpha}{\lambda E(I)} - 1) + \bar{B}(\alpha) \right. \\ & [1 - \frac{\alpha}{\lambda E(I)} - pr\alpha\beta_1 E(V) - p\beta_1\beta_2 E(V) - p\alpha\beta_2 E(V) \\ & + \frac{1}{2} p\alpha\beta_1\beta_2 E(V^2)] \\ & \left. + \bar{B}'(\alpha)[(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) - p\alpha\beta_1\beta_2 E(V)] \right\} \\ & + \lambda Q E(I(I-1)) \{ (r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) \\ & + \bar{B}(\alpha)[p\alpha\beta_1\beta_2 E(V) - (r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha)] \} \end{aligned} \tag{77}$$

$$\begin{aligned} D'(1) = {} & -\lambda E(I)(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) + \bar{B}(\alpha) \{ \alpha\beta_1\beta_2 + \\ & + \lambda E(I)(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) - p\alpha\beta_1\beta_2 E(I) E(V) \} \end{aligned} \tag{78}$$

$$\begin{aligned} D''(1) = {} & 2[\lambda E(I)]^2 \left\{ (1 - \frac{r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha}{\lambda E(I)}) + \bar{B}(\alpha) \right. \\ & [(-1 - pE(V))(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) - \frac{1}{2}\alpha\beta_1\beta_2 E(V^2)] \\ & \left. + \bar{B}_1'(\alpha)[-\frac{\alpha\beta_1\beta_2}{\lambda E(I)} - (r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) + \alpha\beta_1\beta_2 p E(V)] \right\} \\ & + \lambda E(I(I-1)) \{ -(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) \\ & + \bar{B}(\alpha)[(r\alpha\beta_1 + \beta_1\beta_2 + \beta_2\alpha) - \alpha\beta_1\beta_2 p E(V)] \} \end{aligned} \tag{79}$$

where $E(V^2)$ is the second moment of the vacation time and Q has been found in equation (73). Then if we substitute the values of $N'(1), N''(1), D'(1)$ and $D''(1)$ from equations (76), (77), (78) and (79) in to (75) equation we obtain L_q in a closed form.

Mean waiting time of a customer could be found

$$W_q = \frac{L_q}{\lambda} \tag{80}$$

by using Little's formula.

6 Concluding Remarks

We have investigated a single server with Bernoulli scheduled vacation and random break down. When the server is under repair, the first essential repair is followed by second optional repair. The probability generating function of transient solutions are obtained explicitly and along with this the steady state has also been analyzed. Further performance measures like average number of customers in the queue and the average waiting time of a customer in the queue are obtained.

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