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Bayesian Estimation of Structure Variables in the Collective Risk Model for Reserve Risk

Alessandro Ricotta[1] and Edoardo Luini[1]

Abstract

Reserve risk represents a fundamental component of underwriting risk for non-life insurers and its evaluation can be achieved through a wide range of stochastic approaches, including the Collective Risk Model. This paper, in order to fill a gap in existing literature, proposes a Bayesian technique aimed at evaluating the standard deviation of structure variables embedded into the Collective Risk Model. We adopt uninformative prior distributions and the observations of the statistical model are obtained making use of Mack's formula linked to bootstrap methodology. Moreover, correlation between structure variables is investigated with a Bayesian method, where a dependent bootstrap approach is adopted. Finally, a case study is carried out: the Collective Risk Model is used to evaluate the claims reserve of two non-life insurers characterized by a different reserve size. The claims reserve distribution is examined with respect to the total run-off and the one-year time horizon, enabling the assessment of the reserve risk capital requirement.

JEL classification numbers: G22, C63
Keywords: stochastic claims reserving, collective risk model, structure variables, Bayesian approach, bootstrap

1 Introduction

Stochastic claims reserving models allow the assessment of the standard deviation or the probability distribution of claims reserve necessary to quantify the capital charge from a solvency point of view [1]. A variety of stochastic methodologies

[1] Department of Statistics, La Sapienza University of Rome, Italy.

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exist in literature. Mack proposed a first approach [2], [3], [4], which provides the prediction variance related to Chain-Ladder estimate; the variability of the reserve is herein split into Process Variance and Estimation Variance. Furthermore, other methodologies like Bootstrap [5], [6] and Generalized Linear Models [7] are used to determine the claims reserve distribution. In recent years, Bayesian methods have become increasingly important and adopted in stochastic claims reserving; in this paper we follow this line of research with the aim to assess the structural risk factors embedded into the Collective Risk Model to stochastically evaluate the claims reserve. The main advantages of Bayesian models consist in the possibility to investigate distributions of model parameters and the chance to include external information rigorously into actuarial models. In [6] the authors showed that when it comes to incorporating judgment on parameters/parameter distributions underlying a particular statistical model or combining together several statistical models, the Bayesian reserving approach is the preferred option compared to other stochastic reserving methods like the bootstrapping technique [8]. Without being exhaustive, the principal deterministic methods developed under the Bayesian framework are Chain-Ladder [9], [10], [11], Bornhuetter-Ferguson [9], [10], [12] and Overdispersed Poisson Model [12]. Additionally, in [13] different Bayesian approaches to estimate claim frequency are presented and in [14], [15], [16] and [17] it is shown a range of other Bayesian models for both incurred and paid loss data. Furthermore, [18] developed a Bayesian Collective Risk Model where the structure of parameters is based on the deterministic method called Cape Code; the expected loss ratio and the incremental paid loss development factor, which represent model parameters, are evaluated in a Bayesian manner.

The Collective Risk Model (CRM) to assess claims reserve was proposed by different authors (see [18], [19], [20] and [21]). This approach was extended by Ricotta and Clemente [22] assuming that incremental payments to be estimated in the run-off triangle are a compound mixed Poisson process, where the uncertainty on claim size is introduced with a multiplicative structure variable. The model considers, therefore, structure variables on claim count and claim size in order to describe parameter uncertainty on both random variables. In addition, linear dependence between different development and accidental years is addressed.

Literature lacks methodologies designed to calibrate structural risk factors embedded into Collective Risk Theory models for reserve risk. The aim of this paper is to propose a Bayesian procedure to estimate the standard deviation of the structure variables related to the Collective Risk Model as described in [22]. We developed an approach based on two established and widely used methodologies in literature such as the bootstrap method applied to the Chain-Ladder algorithm [7] and the Mack's formula [2]. In addition, the dependence between model parameters, i.e. claim count and claim size, caused by the deterministic average cost method is taken into account; linear correlation, evaluated according to the Bayesian framework, is introduced in the CRM through structural risk factors. Concerning the Bayesian approach adopted to quantify the standard deviation of structure variables, the bootstrap methodology jointed to the Mack's formula is used to enforce the likelihood function. Run-off triangles of different accounting years are considered with the aim to acquire all the accessible historical information available to the insurance company. It is noted that [23] presented a Bayesian bootstrap scheme embedded within an approximate Bayesian computation (ABC) framework to obtain posterior distribution of the Distribution-free Chain-Ladder model parameters and the associated reserve risk measures. In the present paper, instead, the bootstrap procedure, joined to Mack's formula, is adopted to generate the data used to evaluate the likelihood of Bayes' formula. On the other hand, the Bayesian method applied to evaluate correlation between structure variables is built on Mack's formula joined to a dependent bootstrap approach. The bootstrap methodology is herein carried out by jointly resampling in a dependent manner the data into the run-off triangles of claim count and average claim cost, namely entries that fill the same position in the respective run-off triangles. For the estimation of both the standard deviation and the correlation between structure variables, we considered (improper) flat priors over $(0,+\infty)$ and Jeffreys priors. Both of these represent the case when no a priori information is available and the prior is to have minimal influence on the inference; the uniform follows the Laplace postulate or principle of insufficient reason, whereas Jeffreys prior is based on the Fisher information and, as opposed to the former, satisfies the invariant reparametrization requirement [24]. A formalization and discussion of uninformative and improper priors can be found in [25] and [26].

Model parameters different from the structure variables are calibrated by using a data set of individual claims and an average cost method; the deterministic Frequency-Severity method, based on the Chain-Ladder mechanics, is adopted to separately calculate the number of claims and the average costs for each cell of the bottom part of the run-off triangle. Monte Carlo method is performed to simulate the claims reserve distribution according to the whole lifetime of insurer obligations. Furthermore, with regards to a one-year time horizon evaluation, we adapt the "re-reserving" method [27], [28] and estimate both the uncertainty of claims development result and the reserve risk capital requirement.

The paper is organized as follows. Section 2 introduces the Collective Risk Model and displays how to estimate parameters other than structure variables. In Section 3, the Bayesian approach is presented and performed to estimate the standard deviation of structural risk factors; at the same time we report results acquired according to the Metropolis-Hastings algorithm with respect to two non-life insurers. Moreover, the exact moments of structure variables are acquired. Section 4 refers to Pearson correlation coefficient between structural risk factors; results related to the two data sets are also reported. A case study on two non-life insurers is shown in Section 5 where the Collective Risk Model is enforced to evaluate claims reserve distribution concerning both a total run-off and a one-year time horizon. In addition, we investigate the effect of linear correlation magnitude between structure variables on both claims reserve and the average Pearson correlation coefficient affecting outstanding claims of different accident and development years. Conclusions follow.

2 Collective Risk Model

This section reports the main features of the Collective Risk Model developed in [22]. This model, based on the Collective Risk Theory, aims to assess the claims reserve in a stochastic way. Here the claims reserve is represented through the run-off triangle: available data is reported in rectangular table of dimension $N \times N$ where rows $(i = 1,...,N)$ represent the claims accident years (AY), whereas columns $(j = 1,...,N)$ are the development years (DY) related to the number or the amount of claims. Data linked to observed incremental payments fill the upper triangle $D = \{X_{i,j}; i + j \leq N + 1\}$, where $X_{i,j}$ denotes incremental payments of claims in the cell (i, j), namely claims incurred in the generic accident year i and paid after $j-1$ years of development. Analogously, the observed number of claims $n_{i,j}$ in the upper triangle is defined as $D^n = \{n_{i,j}; i + j \leq N + 1\}$. Future numbers or amounts of payments must be assessed for each cell of the lower triangle. The scope is to investigate the random variable[2] (r.v.) of future incremental payments $\tilde{X}_{i,j}$. The CRM represents incremental payments for each cell to be estimated as follows:

$$\tilde{X}_{i,j} = \sum_{h=1}^{\tilde{K}_{i,j}} \tilde{p}\tilde{Z}_{i,j,h}$$

and the r.v. claims reserve, denoted by $\tilde{R}$, is equal to the sum of the cells of lower run-off triangle:

$$\tilde{R} = \sum_{i=1}^{N} \sum_{j=N-i+2}^{N} \tilde{X}_{i,j} \,,$$

where:

- $\tilde{K}_{i,j}$ represents the r.v. number of claims related to the accident year i and paid after $j-1$ years. This r.v. is assumed to be a mixed Poisson process; parameter uncertainty is addressed through a multiplicative structure variable $\tilde{q}$ with unitary mean and standard deviation $\sigma_{\tilde{q}}$. Therefore, the r.v. claims number is parametrized as follow, $\tilde{K}_{i,j} \sim Po(\tilde{q}n_{i,j})$.

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[2] A tilde superscript will henceforth denote random variables.

• $\tilde{Z}_{i,j,h}$ is the random variable describing the amount of the h-th claim occurred in the accident year i and paid after $j-1$ years.

• $\tilde{p}$ denotes the parameter uncertainty related to claim size. This structure variable has mean and standard deviation equal to 1 and $\sigma_{\tilde{p}}$ respectively.

The two structure variables enable the introduction of parameter uncertainty without affecting the expected value of claim number and amount. Furthermore, in the bottom part of the run-off triangle only one r.v. affects the claim number and the claim size respectively, allowing for the dependence between these random variables of different AY and DY given by the settlement process. The assumptions underlying the CRM are the following:

• claim number ($\tilde{K}_{i,j}$), claim cost ($\tilde{Z}_{i,j,h}$), and the structure variable $\tilde{p}$ are mutually independent in each cell (i,j) of the lower run-off triangle;

• claim size values in different cells of the lower run-off triangle are independent and in the same cell are independent and identically distributed (i.i.d.);

• structure variable $\tilde{q}$ is independent of the claim costs in each cell;

• $\tilde{q}$ and $\tilde{p}$ are mutually independent.

In [22] the exact expressions of mean, standard deviation (SD) and skewness of the claims reserve distribution was obtained. The authors showed that the expected value corresponds to the claims reserve estimated by the underlying deterministic method (in our context the Frequency-Severity) and they exhibit the non-negligible impact, on the claims reserve distribution, of structure variables, which turn to be a systemic risk that cannot be diversified by a larger portfolio. Finally, the authors stressed the importance of the estimation of structural risk factors in the CRM; differently to what they proposed, in this paper we developed a Bayesian approach to address this matter.

In order to apply the CRM we need to estimate a set of parameters for each cell (i,j) of the lower triangle. The expected number of paid claims ($n_{i,j}$) and the expected claims cost ($m_{i,j}$) are obtained, conditionally to the set of information D (the run-off triangle of incremental payments) and D^n (the run-off triangle of incremental number of paid claims), with a deterministic average cost method. We use the Frequency-Severity method by applying the Chain-Ladder mechanics on the triangles of cumulative numbers and cumulative average costs[3]. The other quantities necessary to implement the CRM are the cumulants of the severity. According to the claims data set, we estimate the variability coefficient of the claim size for each DY; later, adopting a distribution assumption, the moments of

[3] We adopt the same run-off triangles used in [22].

the r.v. $\tilde{Z}_{i,j}$ are obtained.

3 Bayesian Approach to Estimate the Standard Deviation of Structure Variables

In classical statistics the parameters of a model are assumed to be fixed; Bayesian statistics contrasts with this approach and considers parameters to be random variables (an exhaustive dissertation of the topic can be found in [29], [30] and [31]). The aim of the Bayesian approach is to take parameters uncertainty into account; this variability is introduced through prior probability distributions that, jointly with observed data, allow the posterior probability distribution of the model parameters to be achieved. According to the Bayes theorem, the parameter posterior distribution, $f(\theta \mid x)$, can be computed as:

$$f(\theta \mid x) = \frac{f(x \mid \theta) f(\theta)}{f(x)}.$$

The term $f(x \mid \theta)$ is the sampling density of data under a chosen probability model; this element, viewed as function of θ for fixed x, is the likelihood function. The parameter prior distribution is $f(\theta)$, which refers to the parameter uncertainty, also interpretable as the prior opinion or knowledge related to parameter values. The denominator of Bayes' formula represents the marginal distribution of data. This quantity does not depend on θ and with fixed x turns out to be a constant quantity which acts as a normalizing factor that leads to a proper posterior distribution. Bayes theorem is often considered without the normalizing constant that has only the effect of rescaling the density:

$$f(\theta \mid x) \propto f(x \mid \theta) f(\theta).$$

Hence, the posterior distribution is proportional to the product of likelihood function and prior. Therefore, Bayes' formula depends on data and prior distribution. Typically, prior distributions are classified as uninformative and informative distributions. The former ideally refers to the principle of indifference and is typically flat distributions that assign equal probability to all possible values of the parameter, with the aim to have a minimal effect, relative to the data, on the posterior inference. On the other hand, informative distributions are calibrated using observed data. Bayes' approach also allows us to make inference on future observation through the posterior predictive distribution, where the adjective posterior refers to the consideration that the distribution is conditional to the observed data (x), and predictive because it is a prediction of new observable data (y). The posterior predictive distribution is an average of the probability distribution of y conditional on the unknown value of θ, weighted with the posterior distribution of θ:

$$f(y \mid x) = \int f(y \mid \theta) f(\theta \mid x) d\theta .$$

Hence, outcomes of the Bayesian analysis are the posterior predictive distribution, which provides information about new observations, and the posterior distribution, which contains information about the parameters underlying the model. With regards to the posterior distribution, it is possible to summarize this information by developing different types of inference analysis on this distribution (i.e. both point or region estimation and hypothesis testing).

The Bayesian framework here is used to calibrate the standard deviation of structure variables of CRM. These variables related to claim count and claim cost do not affect the expected value of the reserve but have an impact on the other characteristics (i.e. variance, skewness and so on). As adopted in [22], we follow the usual assumption of Collective Risk Theory that structure variables are gamma distributed with identical parameters:

$$\tilde{q} \sim Gamma(h;h), \quad \tilde{p} \sim Gamma(k;k).$$

The variables $\tilde{q}$ and $\tilde{p}$ have mean equal to 1, given by the ratio of the parameters, and standard deviation $\sigma_{\tilde{q}} = 1/\sqrt{h}$ and $\sigma_{\tilde{p}} = 1/\sqrt{k}$. Therefore, the values of $\sigma_{\tilde{q}}$ and $\sigma_{\tilde{p}}$ determine the parameter of interest, h and k, which all the characteristics of the structure variable depend upon. In [22] a deterministic approach based on the Estimation Variance derived via Mack is proposed to assess the parameters of structure variables. In Mack's formula, the Estimation Error measures the variability produced by the parameters estimation; because of this, it is ascribable to the structure variables that have the aim to introduce parameters uncertainty on quantities being considered (i.e. claim count and severity). Here the standard deviations $\sigma_{\tilde{q}}$ and $\sigma_{\tilde{p}}$ are interpreted as random variables and consequently later denoted by a tilde (random variables and their parameters are denoted with the subscript $\tilde{q}$ or $\tilde{p}$ to indicate which r.v. is considered in the Bayes approach, whereas if general considerations are carried out, the subscript is omitted for a simpler notation). It is assumed that, $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$, define for positive values, follow a gamma distribution:

$$\tilde{\sigma} \sim Gamma\left(\tilde{A};\tilde{B}\right),$$

where the parameters $\tilde{A}$ and $\tilde{B}$ are random variables with regards to prior information is conveyed. Parameters of $\tilde{A}$ and $\tilde{B}$ are called hyperparameters of the model. In this context, the evaluation of the standard deviation of structure variables is acquired through the Bayes' formula with the purpose to obtain a posterior distribution of the parameters which $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ depend on:

$$f(A,B \mid \tilde{\sigma}) \propto f\left(\sigma \mid \tilde{A},\tilde{B}\right) f(A) f(B) \tag{3.1}$$

With regards to the posterior distributions achieved via the Bayesian method, their expected values are used to calibrate the random variables (r.v.s) $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$. Therefore, the posteriors means are adopted to estimate the parameters of the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$:

$$\tilde{\sigma} \sim Gamma\left(E\left(\tilde{A} \mid \tilde{\sigma}\right); E\left(\tilde{B} \mid \tilde{\sigma}\right)\right) \tag{3.2}$$

It may be noted that, as depicted in formula (3.1) above, we are assuming $\tilde{A}$ and $\tilde{B}$ prior probability distributions to be independent; this premise is however neither affecting nor restrictive on our model for two reasons. First and foremost, since only the posterior expected values of $\tilde{A}$ and $\tilde{B}$ enter formula (3.2), we are looking separately at the marginal posterior distributions of the parameters (when computing one parameter expectation, the other one is automatically marginalized out). Secondly, in the outlined framework, even if starting with independent priors, the Bayes theorem formula will generate a dependent posterior distribution, whose dependency is induced by the likelihood function.

The likelihood function of formula (3.1) is implemented making use of Mack's formula and bootstrap methodology. The latter is carried out following the procedure adopted in [7]. Within the Chain-Ladder framework, the bootstrap method, by resampling the upper triangle of model residuals, allows us to create different resampled data sets, which can be used to calculate the quantity of interest and make inference on it. For our purposes, we applied the bootstrap approach to the run-off triangles of the cumulative claim count and cumulative average cost. On every iteration, for both triangles, the square root of the Estimation Variance derived via Mack's formula is divided by the respective Chain-Ladder estimate (i.e. the mean of frequency and severity) with the aim to measure the variability produced by the parameters estimation. These relative variabilities, concerning only the Estimation Error, are interpreted as the coefficient of variation of the structure variables $\tilde{q}$ and $\tilde{p}$; bearing in mind that their means are equal to 1, these values correspond to the standard deviations $\sigma_{\tilde{q}}$ and $\sigma_{\tilde{p}}$ and are interpreted as the uncertainty related to the parameters estimate. The quantities $\sigma_{\tilde{q}}$ and $\sigma_{\tilde{p}}$ are written here without tilde because they represent one generic realization of the corresponding r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$. In detail, each simulation step of the Mack-Bootstrap procedure consists of the following stages.

1. Determine the Chain-Ladder development factors, the so-called link ratios, for each development year according to the observed data in the upper run-off triangle.

2. From the link ratios and the data observed in the last available diagonal of the triangle, recursively calculate cumulative amounts in the upper run-off triangle, and then incremental data by subtraction.

3. Compute the adjusted Pearson's residuals of the model from incremental data obtained in the previous step and the original observed data of the run-off triangle.

4. By sampling the residuals with replacement, create the run-off triangle of residuals and, from this, achieve cumulative data.

5. Enforce Chain-Ladder method and Mack's formula to estimate quantities of interest: the ratio between the square root the Estimation Variance and the Chain-Ladder estimate provides the standard deviation of the structure variable under analysis, namely the standard deviation of either $\tilde{q}$ or $\tilde{p}$.

We thus estimate $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ sampling distributions, which will be leveraged to implement the Bayes' theorem, by making use of two sound methodologies: the bootstrap scheme applied to the Chain-Ladder algorithm [7] and the Mack's formula [2]. The bootstrap procedure joined to the Mack's formula takes as input the run-off triangles of both claim count and average costs of different dimensions; the aim is to take into consideration all available historical information. Starting from the run-off triangle related to the current accounting year, with dimension $N \times N$, this approach is performed on triangles obtained by gradually deleting the last diagonal available, one at a time. Therefore, the run-off triangles of different accounting years are considered up to the current triangle, where the first run-off triangle is chosen starting from the earliest information considered currently representative. Hence, in respect to the last *l* accounting years, the run-off triangles have dimensions $\left(N-l+1\times N-l+1\right),...,\left(N\times N\right)$ respectively and the Mack's formula applied to the bootstrap scheme lets us obtain for each historical triangle the sample distribution of random variables $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$. The likelihood functions $f\left(\sigma_{\tilde{q}} \mid \tilde{A}_{\tilde{q}}, \tilde{B}_{\tilde{q}}\right)$ and $f\left(\sigma_{\tilde{p}} \mid \tilde{A}_{\tilde{p}}, \tilde{B}_{\tilde{p}}\right)$, based on the gamma model, are evaluated at the expected values of the distributions of $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ related to the sequence of the *l* historical triangles. Thus, concerning a generic historical triangle, the sample mean of the distribution of the standard deviation, namely the average variability of parameter estimation that affects the triangle, is adopted as an estimate of the true unobservable historical value of the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$. The *l* values of $E\left(\tilde{\sigma}_{\tilde{q}}\right)$ and $E\left(\tilde{\sigma}_{\tilde{p}}\right)$ are interpreted as data and used to compute likelihood functions; we assume this data to be independent and identically distributed. However, it is to be noted that the latter assumption, useful to calculate the likelihood, in practice does not fully hold since data is attained on the *l* historical triangles that share common cells, affecting the assumption of independence; moreover, the model is lacking in conditions apt to fulfill the identical distribution assumption of data.

We consider uninformative priors related to the positive parameters of $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ with the aim to prevent any sort of expert judgment. In particular, $f\left(A\right)$ and $f\left(B\right)$ are modeled either via uniform or Jeffreys distributions. The form of the Jeffreys prior depends on the likelihood model selected, and its functional dependence on the likelihood is invariant under reparameterization of the

parameter. Jeffreys priors for two-parameter gamma distribution are easily derived from [32].
The Bayesian method, as described above, has been applied to the claim data sets of two non-life insurance companies working in the Motor Third Party Liability (MTPL) line of business and concerning accounting years from 1993 to 2004. DELTA insurer is a small-medium company, whereas OMEGA insurer is roughly 10 times larger. Appendix A reports the run-off triangles adopted to estimate, via the Frequency-Severity deterministic method, the claims reserve. Triangles related to cumulative claim count and cumulative average costs are used to enforce the Bayesian methodology detailed above. The bootstrap joined to the Mack's formula has been carried out regarding run-off triangles for 9 accounting years; therefore, the triangles adopted to acquire historical data and calibrate prior distributions have dimension from 4×4 to 12×12. The number of iterations carried out in the bootstrap stage is equal to 10,000. In respect to the insurer DELTA, the distributions of $\tilde{\sigma}_{\tilde{q}}$ acquired via the Mack-Bootstrap procedure related to the 9 historical triangles of claim count show expected values included between 1.66% and 2.48%; for $\tilde{\sigma}_{\tilde{p}}$ the minimum value of the mean is 2.33% whereas the maximum is 4.29%. Table 1 details the expected values, 5% quantile and 95% quantile related to the distributions of $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$.

Table 1: DELTA - Expected value, 5% quantile and 95% quantile of the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ related to the historical triangles with dimensions from 4×4 to 12×12.

DELTA						
Dimension	$\tilde{\sigma}_{\tilde{q}}$			$\tilde{\sigma}_{\tilde{p}}$		
	Exp.Value	**Quantile 5%**	**Quantile 95%**	**Exp.Value**	**Quantile 5%**	**Quantile 95%**
4x4	1.83%	0.66%	3.37%	4.15%	1.11%	9.77%
5x5	1.66%	0.82%	2.82%	4.29%	1.78%	8.32%
6x6	1.75%	1.08%	2.69%	2.81%	1.36%	5.09%
7x7	2.30%	1.32%	3.58%	2.72%	1.47%	4.78%
8x8	2.48%	1.66%	3.64%	2.93%	1.84%	4.65%
9x9	2.40%	1.66%	3.45%	2.68%	1.74%	4.24%
10x10	2.32%	1.72%	3.21%	2.66%	1.81%	4.04%
11x11	2.42%	1.84%	3.28%	2.67%	1.89%	3.93%
12x12	2.26%	1.76%	3.01%	2.33%	1.72%	3.23%

With respect to the larger insurer OMEGA, the Mack-Bootstrap procedure leads to expected values between 2.04% and 4.94% for $\tilde{\sigma}_{\tilde{q}}$, and between 2.34% and

3.16% for $\tilde{\sigma}_{\tilde{p}}$. Table 2 depicts the means and quantiles of order 5% and 95% regarding the distributions of the two random variables.

Table 2: OMEGA - Expected value, 5% quantile and 95% quantile of the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ related to the historical triangles with dimensions from 4×4 to 12×12.

OMEGA						
Dimension	$\tilde{\sigma}_{\tilde{q}}$			$\tilde{\sigma}_{\tilde{p}}$		
	Exp.Value	**Quantile 5%**	**Quantile 95%**	**Exp.Value**	**Quantile 5%**	**Quantile 95%**
4x4	4.94%	1.51%	9.38%	2.38%	0.62%	5.42%
5x5	3.29%	1.32%	5.92%	3.16%	1.30%	6.29%
6x6	2.41%	1.06%	4.15%	2.93%	1.46%	5.28%
7x7	2.25%	1.24%	3.57%	3.02%	1.63%	5.27%
8x8	2.04%	1.23%	3.10%	2.62%	1.48%	4.57%
9x9	2.04%	1.33%	2.98%	2.67%	1.59%	4.47%
10x10	2.36%	1.52%	3.52%	2.38%	1.49%	3.84%
11x11	2.80%	1.88%	4.20%	2.70%	1.67%	4.43%
12x12	2.98%	2.14%	4.26%	2.34%	1.61%	3.48%

Having chosen the prior distributions, the next step is to calculate the posteriors. Since the posterior distributions being examined do not have a closed-form expression, we make use of the Metropolis-Hastings algorithm to draw samples from them. For generating a sample (commonly referred to as chain) from the posterior distribution, this Markov Chain Monte-Carlo method requires only a function that is proportional to the real density, rather than exactly equal to it, avoiding the calculation of the normalization factor, which is extremely difficult in practice, especially when dealing with multi-dimensional distributions [33]. In particular, a Random Walk Metropolis algorithm has been selected; this version of the Metropolis-Hastings design operates by proposing that the chain move to a candidate state obtained by disturbing the current one with a noise. Under mild conditions the chain converges to its stationary distribution and posterior quantities can be estimated from the simulation output. A comprehensive dissertation of the topic can be found in [34].

Concerning the posterior distributions achieved via the above-mentioned algorithm, we use, for our purposes, the expected values to assess the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$, whose parameters are set equal to the means of the posteriors, as shown in formula (3.2). It is to be noted that, for both insurers, the two kind of uninformative prior distributions (uniform and Jeffreys) lead to similar results in terms of expected values of the posteriors (see Appendix B). Posterior expected values are negligibly affected, in our model, by the distribution type adopted as prior distributions and hence, when the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ are calibrated, their characteristics are not significantly impacted by the prior distribution assumption. Tables 3 and 4 indicate the expected values and coefficient of variations, for both insurers, under uniform and Jeffreys priors.

Table 3: DELTA - Uninformative priors. Expected values and coefficient of variation of $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$.

DELTA				
Random Variable	$\tilde{\sigma}_{\tilde{q}}$		$\tilde{\sigma}_{\tilde{p}}$	
Type of prior	**Uniform**	**Jeffreys**	**Uniform**	**Jeffreys**
Expected value	2.15%	2.15%	3.02%	3.03%
Coeff. of variation	12.43%	14.35%	17.63%	20.34%

Table 4: OMEGA - Uninformative priors. Expected values and coefficient of variation of $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$.

OMEGA				
Random Variable	$\tilde{\sigma}_{\tilde{q}}$		$\tilde{\sigma}_{\tilde{p}}$	
Type of prior	**Uniform**	**Jeffreys**	**Uniform**	**Jeffreys**
Expected value	2.77%	2.79%	2.69%	2.69%
Coeff. of variation	24.04%	27.69%	9.00%	10.39%

The assessment of the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ allows us to determine the moments of the structure variables $\tilde{q}$ and $\tilde{p}$ adopted into the CRM. Taking into consideration the structural risk factors related to claim count (identical considerations hold for $\tilde{p}$), we assume that $\tilde{q} \sim Gamma\left(\frac{1}{\tilde{\sigma}_{\tilde{q}}^2};\frac{1}{\tilde{\sigma}_{\tilde{q}}^2}\right)$ and $\tilde{\sigma}_{\tilde{q}} \sim Gamma\left[E\left(\tilde{A}_{\tilde{q}}^{post}\right);E\left(\tilde{B}_{\tilde{q}}^{post}\right)\right]$, where $E\left(\tilde{A}_{\tilde{q}}^{post}\right)$ and $E\left(\tilde{B}_{\tilde{q}}^{post}\right)$ represent the expected values of the posterior distributions, $E\left(\tilde{A}_{\tilde{q}} \mid \tilde{\sigma}_{\tilde{q}}\right)$ and $E\left(\tilde{B}_{\tilde{q}} \mid \tilde{\sigma}_{\tilde{q}}\right)$. Moments of the structure variable depend on the parameters of the mixing variable $\tilde{\sigma}_{\tilde{q}}$, which however does not affect the mean of $\tilde{q}$ that remains equal to 1 (see Appendix C for details). The variance is described by the following formula:

$$Var\left(\tilde{q}\right)=\frac{E\left(\tilde{A}_{\tilde{q}}^{post}\right)\left[E\left(\tilde{A}_{\tilde{q}}^{post}\right)+1\right]}{E\left(\tilde{B}_{\tilde{q}}^{post}\right)^{2}}.$$

The coefficient of variation, equal to the square root of variance, is:

$$CV\left(\tilde{q}\right)=\frac{\sqrt{E\left(\tilde{A}_{\tilde{q}}^{post}\right)\left[E\left(\tilde{A}_{\tilde{q}}^{post}\right)+1\right]}}{E\left(\tilde{B}_{\tilde{q}}^{post}\right)}.$$

Finally, the skewness of the structure variable is given by:

$$\gamma\left(\tilde{q}\right)=\frac{2\left[E\left(\tilde{A}_{\tilde{q}}^{post}\right)+2\right]\left[E\left(\tilde{A}_{\tilde{q}}^{post}\right)+3\right]}{E\left(\tilde{B}_{\tilde{q}}^{post}\right)\sqrt{E\left(\tilde{B}_{\tilde{q}}^{post}\right)\left[E\left(\tilde{B}_{\tilde{q}}^{post}\right)+1\right]}}.$$

Tables 5 and 6 report the exact characteristics for the structure variable $\tilde{q}$ and $\tilde{p}$ for both the insurers:

Table 5: DELTA - Expected value, coefficient of variation and skewness related to structure variables $\tilde{q}$ and $\tilde{p}$.

DELTA				
Type of prior	**Uniform**		**Jeffreys**	
Structure variable	$\tilde{q}$	$\tilde{p}$	$\tilde{q}$	$\tilde{p}$
Expected value	1	1	1	1
Coeff. of variation	2.17%	3.06%	2.18%	3.09%
Skewness	0.046	0.069	0.047	0.072

Table 6: OMEGA - Expected value, coefficient of variation and skewness related to structure variables $\tilde{q}$ and $\tilde{p}$.

OMEGA				
Type of prior	**Uniform**		**Jeffreys**	
Structure variable	$\tilde{q}$	$\tilde{p}$	$\tilde{q}$	$\tilde{p}$
Expected value	1	1	1	1
Coeff. of variation	2.85%	2.70%	2.90%	2.70%
Skewness	0.071	0.056	0.076	0.056

4 Bayesian Estimation of the Pearson Correlation Coefficient between Structure Variables

The Collective Risk Model assumes that claim count and claim size are mutually independent in each cell (i, j) of the lower run-off triangle. However, this theoretical assumption does not hold in practice due to the dependence introduced on model parameters by the average cost method (i.e. Frequency-Severity). The aim of this section is to evaluate, using a Bayesian procedure, the Pearson correlation coefficient between claim count and claim cost, estimated on structure variables $\tilde{q}$ and $\tilde{p}$. It is noteworthy that the procedure we define could be implemented with measures of rank correlation, such as Spearman's rho and Kendall's tau. As opposed to Pearson correlation coefficient, these are able to capture more general monotonic relationships between variables, and thus they can better detect non-linear forms of association. The user should assess which measure of correlation is more appropriate on a case-by-case basis. In our context, having preliminarily analysed different types of correlation measures, we considered linear correlation to be suitable for describing the dependence between structure variables.

Similarly to Section 3, we adopt a method based on bootstrap resampling and Mack's formula, in which, however, the former considers the dependency between the run-off triangles of claim count and average claim cost, by resampling pairs of data which fill the same position in the respective triangles. The scope is to build up the distributions of the r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ by implicitly allowing for the dependence, caused by the average cost method, between the two data sets of claim count and claim cost. Hence, the estimated Pearson correlation coefficient is used to calibrate a Gaussian copula with the purpose to set up a two-dimensional random variable where the marginals are the two r.v.s $\tilde{q}$ and $\tilde{p}$ calibrated in the previous section.

In the Bayesian framework, the Pearson correlation coefficient is interpreted as a random variable following a beta distribution:

$$\rho \sim Beta\left(\tilde{C};\tilde{D}\right).$$

We analysed the dependence between claim count and claim cost on the interval $[0,1]$; therefore, we assume parameter variabilities to be positively correlated. As usual, the r.v.s $\tilde{C}$ and $\tilde{D}$ identify prior distributions. According to Bayes' rule we obtain a posterior distribution of the parameters which $\tilde{\rho}$ depends on:

$$f\left(C,D \mid \tilde{\rho}\right) \propto f\left(\rho \mid \tilde{C},\tilde{D}\right) f\left(C\right) f\left(D\right) \tag{4.1}$$

The expected value of the posterior is used to calibrate the r.v. $\tilde{\rho}$:

$$\tilde{\rho} \sim Beta\left(E\left(\tilde{C} \mid \tilde{\rho}\right);E\left(\tilde{D} \mid \tilde{\rho}\right)\right) \tag{4.2}$$

Finally, the mean of $\tilde{\rho}$, calculated with the posterior expected value, is adopted to assess the Gaussian copula used to join the marginals $\tilde{q}$ and $\tilde{p}$. Likelihood function of (4.1) is performed making use of Mack's formula and the dependent bootstrap approach. The likelihood function based on the beta model is executed using the Pearson correlation coefficient calculated between the distribution of

$\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ related the sequence of the l historical triangles. Priors, as with the analysis of structure variable standard deviation, are either flat or Jeffreys. In the latter case, the prior formulation relies on the beta likelihood model and the relevant definition can be found in [35].

Below are the results of the previous Bayesian approach adopted to estimate correlation between structure variables, concerning the two insurers introduced in Section 3. As usual, the analyses are based on 10,000 iterations carried out with the dependent bootstrap technique. Table 7 exhibits values of Pearson correlation coefficient computed on the 9 historical triangles via the Mack's formula and dependent bootstrap approach. The linear correlation of the small insurer, DELTA, is included between 0.014 and 0.333, whereas OMEGA shows values between 0.075 and 0.415.

Table 7: Pearson correlation coefficient for both insurers between r.v.s $\tilde{\sigma}_{\tilde{q}}$ and $\tilde{\sigma}_{\tilde{p}}$ related to the historical triangles with dimension from 4×4 to 12×12.

Pearson Correlation Coefficient									
Dimension	**4x4**	**5x5**	**6x6**	**7x7**	**8x8**	**9x9**	**10x10**	**11x11**	**12x12**
DELTA	0.231	0.333	0.127	0.014	0.098	0.065	0.154	0.194	0.087
OMEGA	0.289	0.174	0.220	0.415	0.289	0.324	0.175	0.106	0.075

The posterior is achieved via Monte Carlo method through Metropolis-Hasting algorithm; the mean of the posterior (see Appendix D) is used to assess parameters of the r.v. $\tilde{\rho}$ as shown in formula (4.2). Finally, we adopt the expected value of $\tilde{\rho}$ as an estimate of the Pearson correlation coefficient between structure variables $\tilde{q}$ and $\tilde{p}$. Table 8 reports the correlation between structural risk factors estimated under uniform and Jeffreys priors.

Table 8: Uninformative priors: estimated Pearson correlation coefficient between the structure variables for both insurers.

Estimated expected values of $\tilde{\rho}$			
Insurer	**Type of prior**	**Uniform**	**Jeffreys**
DELTA	$E(\tilde{\rho})$	0.139	0.144
OMEGA	$E(\tilde{\rho})$	0.237	0.241

Concerning the Collective Risk Model, the structure variables are modeled with a two-dimensional meta-Gaussian distribution, where a Gaussian copula, with parameter the Pearson correlation coefficient estimated as shown above, joins the two marginals of $\tilde{q}$ and $\tilde{p}$ calibrated as explained in the previous section.

5 Case Study

The estimates related to structure variables acquired in Sections 3 and 4 are deployed here into the Collective Risk Model in order to evaluate the claims reserve distribution concerning both a total run-off and a one-year time horizon. By adapting the re-reserving method we obtain the "one-year" reserve distribution of insurer obligations. Reserve risk is assessed by calculating the Solvency Capital Requirement (SCR) as the difference between the quantile at 99.5% confidence level of the distribution of the insurer obligations at the end of the next accounting year, opportunely discounted at time zero, and the best estimate at present time.
As explained in Section 1, model parameters related to claim size and claim count are estimated through the deterministic Frequency-Severity method (run-off triangles are reported in Appendix A); moreover, to calibrate cumulants of severity we consider the variability coefficient of claim cost for each development year and we assume that $\tilde{Z}_{i,j}$ follows a gamma distribution in each cell of the triangle.
The deterministic method leads DELTA and OMEGA to a claims reserve of approximately 228 and 2,807 million Euro; these values match the expected values (best estimates) attained with the CRM. The analyses shown below are based on 100,000 simulations; moreover, model parameters acquired via Bayesian approaches are based only on uniform priors. Under the assumption of uncorrelated structure variables, we verify that simulated moments of the claims reserve are close to the exact ones, proving that the number of simulations is adequate. Table 9 refers to the two analysed insurers and reports the mean, standard deviation, coefficient of variation and skewness of the claims reserve evaluated both under a total run-off and a one-year time horizon, assuming $\rho(\tilde{q},\tilde{p})=0$.

Table 9: Mean, standard deviation, coefficient of variation and skewness of claims reserve assessed under total run-off and one-year time horizon for both insurers under the assumption of no correlation between $\tilde{q}$ and $\tilde{p}$. Monetary amounts are expressed in thousands of Euro.

Insurer	Time horizon	Mean	Std. dev	Coeff. of Var.	Skewness
DELTA	Tot. run-off	228,389	13,009	5.70%	0.137
	One-year	228,570	11,721	5.13%	0.214
OMEGA	Tot. run-off	2,807,275	117,500	4.19%	0.098
	One-year	2,805,602	84,922	3.03%	0.121

The relative variability of the reserve assessed under a total run-off time horizon is higher compared to the one-year time horizon for both insurers; on the other hand,

the claims reserve is more skewed under a one-year evaluation. It is to be noted that the coefficient of variation of the one-year reserve, compared to the total run-off value, is approximately around the 90% and 70% for DELTA and OMEGA respectively. Comparing the two insurers, the coefficient of variation is lower for OMEGA than for DELTA, due to a bigger number of reserved claims that leads to a higher diversification of pooling risk, namely the variability not ascribable to structure variables. Similarly, the reserve of the bigger insurer is less skewed in respect of the obligations distribution of DELTA.
Table 10 refers to the one-year claims reserve and gives the mean, standard deviation, quantile and Tail VaR at level 99.5% and 99% respectively; moreover, the Solvency Capital Requirement and its ratio respect to the best estimate, the so-called SCR ratio, are reported.

Table 10: One-year reserve: mean, standard deviation, quantile and TVaR at level 99.5% and 99% respectively, SCR and SCR ratio under the assumption of no correlation between $\tilde{q}$ and $\tilde{p}$. Monetary amounts are expressed in thousands of Euro.

Insurer	Mean	Std. dev.	$q_{99.5\%}$	$TVaR_{99\%}$	SCR	SCR ratio
DELTA	228,570	11,721	261,332	262,604	37,762	14.33%
OMEGA	2,805,602	84,922	3,036,234	3,046,723	230,632	8.22%

OMEGA shows a smaller SCR ratio than DELTA, due to lower values of both relative variability and skewness. The larger claims size allows the pooling risk to be diversified with a higher degree: OMEGA indeed displays lower values of standard deviation and skewness than DELTA ones. This aspect leads to a smaller SCR ratio.
In what follows, we investigate the impact that dependence between structure variables has on claims reserve; in addition to the correlation value estimated via the Bayesian approach, we impose perfect negative and positive linear correlations between $\tilde{q}$ and $\tilde{p}$. When considering the claims reserve evaluated according to a total run-off time horizon, under the assumption of no correlation between the structural risk factors, it is possible to calculate the coefficient of variation of the reserve in respect of the average Pearson correlation coefficient ($\bar{\rho}$) affecting the cells of the lower triangle,

$$Var\left(\tilde{R}\right)=\left[\sum_{i,j\in B}Var\left(\tilde{X}_{i,j}\right)\right]\left(1-\bar{\rho}\right)+\bar{\rho}\left[\sum_{i,j\in B}SD\left(\tilde{X}_{i,j}\right)\right]^2,$$

where, to simplify the notation, $B=\left\{\tilde{X}_{i,j};i+j>N+1\right\}$ identifies cells of the lower run-off triangle. It is worth noting that the average Pearson correlation coefficient affecting $n\geq 3$ random variables has the lower bound [36]:

$$\bar{\rho}_{\min} \geq -\frac{1}{n-1}$$

In the triangle of dimension 12×12 the number of lower cells is 66: this leads to a theoretical value of $\bar{\rho}_{\min}$ equal to -0.015. It is worth emphasising that insurer DELTA shows higher values of relative variability being equal values of $\bar{\rho}$. Later, through simulation, the coefficient of variation of the reserve is calculated using the correlation between structure variables estimated via the Bayesian approach, and values ± 1. This allows us to indirectly quantify the equivalent average Pearson correlation coefficient induced in the cells of the triangle under the assumption of no correlation between the r.v.s $\tilde{q}$ and $\tilde{p}$.

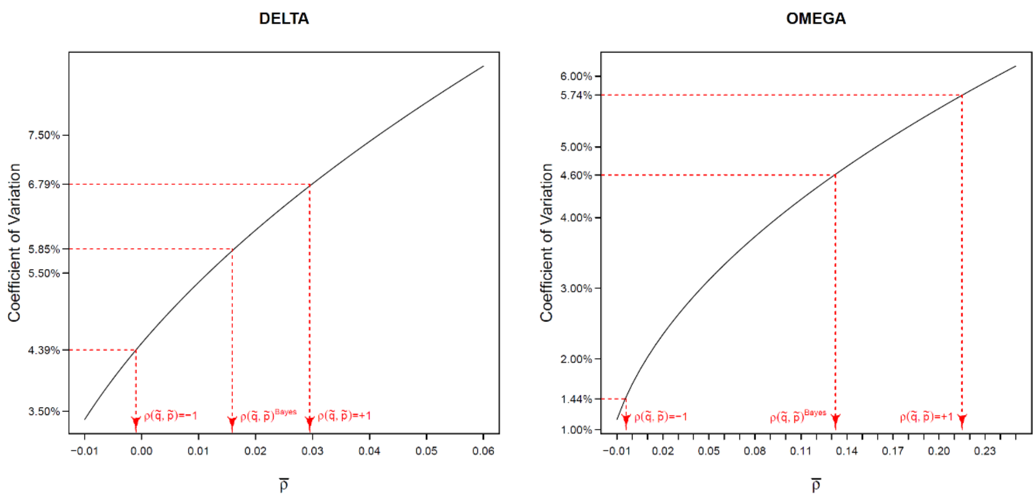

Figure 1: Average Pearson correlation coefficient induced by the dependence between structure variables.

The impact that the dependence between structure variables has on $\bar{\rho}$ is higher for the larger insurer: indeed the claims reserve distribution of OMEGA is affected mainly by the structure variables, due to its higher number of reserved claims which allows the pooling risk to be almost entirely diversified. According to the previously considered values of Pearson correlation between $\tilde{q}$ and $\tilde{p}$, Tables 11 and 12 compare the characteristics of the total run-off and a one-year reserve.

Table 11: DELTA - Mean, standard deviation, coefficient of variation and skewness for both total run-off and one-year claims reserve for different levels of dependence between $\tilde{q}$ and $\tilde{p}$. Monetary amounts are expressed in thousands of Euro.

DELTA					
Perason correlation coeff.	**Time horizon**	**Mean**	**Std. dev.**	**Coeff. of Var.**	**Skewness**
$\rho(\tilde{q},\tilde{p}) = -1$	Tot. run-off	228,192	10,014	4.39%	0.116
	One-year	228,409	10,078	4.41%	0.194
$\rho(\tilde{q},\tilde{p}) = 0.139$	Tot. run-off	228,437	13,359	5.85%	0.137
	One-year	228,588	11,922	5.22%	0.233
$\rho(\tilde{q},\tilde{p}) = +1$	Tot. run-off	228,568	15,514	6.79%	0.162
	One-year	228,720	13,135	5.74%	0.238

Table 12: OMEGA - Mean, standard deviation, coefficient of variation and skewness for both total run-off and one-year claims reserve for different levels of dependence between $\tilde{q}$ and $\tilde{p}$. Monetary amounts are expressed in thousands of Euro.

OMEGA					
Perason correlation coeff.	**Time horizon**	**Mean**	**Std. dev.**	**Coeff. of Var.**	**Skewness**
$\rho(\tilde{q},\tilde{p}) = -1$	Tot. run-off	2,804,619	40,291	1.44%	0.005
	One-year	2,803,575	45,314	1.62%	0.073
$\rho(\tilde{q},\tilde{p}) = 0.237$	Tot. run-off	2,807,968	129,231	4.60%	0.109
	One-year	2,805,983	91,958	3.28%	0.133
$\rho(\tilde{q},\tilde{p}) = +1$	Tot. run-off	2,809,701	161,137	5.74%	0.154
	One-year	2,807,209	111,329	3.97%	0.184

Moreover, taking into consideration the distribution of insurer obligations at the end of the next accounting year, Tables 13 and 14 report the mean, standard deviation, some risk measures (i.e. quantile and Tail VaR at 99.5% and 99% confidence level respectively), the Solvency Capital Requirement and the SCR ratio. Figure 2 refers to the claims development result distribution; this distribution, by construction, has mean equal to zero since it is obtained as difference between the distribution of insurer obligations at the end of the next accounting year and the current expected value of the claims reserve.

Table 13: DELTA - One-year reserve: mean, standard deviation, quantile and TVaR at level 99.5% and 99% respectively, SCR and SCR ratio for different levels of dependence between structure variables. Monetary amounts are expressed in thousands of Euro.

DELTA						
Perason correlation coeff.	**Mean**	**Std. dev.**	$q_{99.5\%}$	$TVaR_{99\%}$	**SCR**	**SCR ratio**
$\rho(\tilde{q},\tilde{p}) = -1$	228,409	10,078	256,332	257,386	27,923	12.23%
$\rho(\tilde{q},\tilde{p}) = 0.139$	228,588	11,922	262,193	263,562	33,605	14.70%
$\rho(\tilde{q},\tilde{p}) = +1$	228,720	13,135	265,905	267,515	37,186	16.26%

Table 14: OMEGA - One-year reserve: mean, standard deviation, quantile and TVaR at level 99.5% and 99% respectively, SCR and SCR ratio for different levels of dependence between structure variables. Monetary amounts are expressed in thousands of Euro.

OMEGA						
Perason correlation coeff.	**Mean**	**Std. dev.**	$q_{99.5\%}$	$TVaR_{99\%}$	**SCR**	**SCR ratio**
$\rho(\tilde{q},\tilde{p}) = -1$	2,803,575	45,314	2,922,753	2,927,557	119,178	4.25%
$\rho(\tilde{q},\tilde{p}) = 0.237$	2,805,983	91,958	3,059,477	3,069,512	253,494	9.03%
$\rho(\tilde{q},\tilde{p}) = +1$	2,807,209	111,329	3,125,179	3,140,281	317,970	11.33%

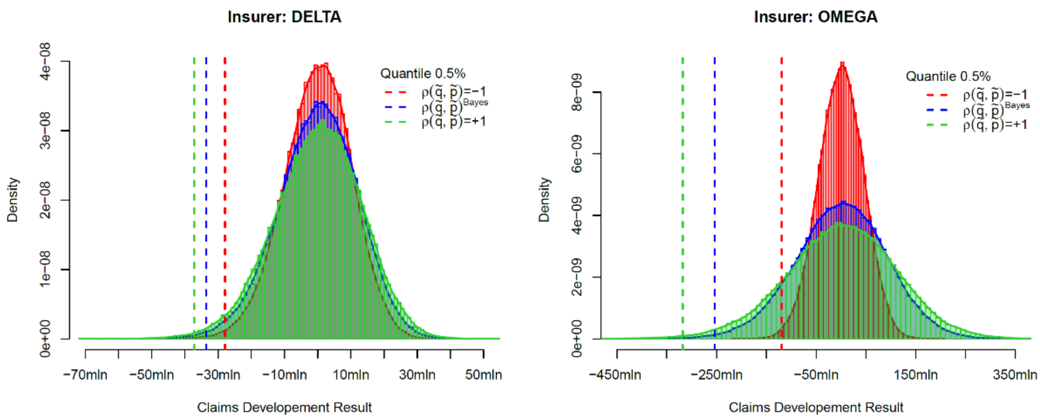

Figure 2: One-year time horizon: distribution of the Claims Development Result for both insurers.

Both insurers show an increasing value of the SCR ratio in respect of the linear correlation between structure variables. When dependence between structural risk factors augments, the standard deviation and the quantile at level 99.5% of the claims reserve increase, leading to a greater solvency capital charge. It is to be pointed out that for both insurers a value of linear correlation between $\tilde{q}$ and $\tilde{p}$ equal to -1 leads to a particular situation where the coefficient of variation of the one-year reserve is higher than the relative variability of the total run-off distribution. The ratio between the coefficients of variation of the one-year reserve and the total run-off reserve is 100.55% and 112.51% for DELTA and OMEGA respectively.

To investigate this unique circumstance we focus on the variance of the reserve evaluated under the two time horizons (since the one-year and the total run-off distributions have the same mean) expressed in terms of the average Pearson correlation coefficient:

$$\begin{aligned}&\sum_{i,j\in B} Var\left(\tilde{X}_{i,j}^{OY}\right)+\bar{\rho}^{OY}\sum_{i,j\in B}\sum_{\substack{h,k\in B\\(h\neq i\,\vee\,k\neq j)}} SD\left(\tilde{X}_{i,j}^{OY}\right)SD\left(\tilde{X}_{h,k}^{OY}\right)\\&>\sum_{i,j\in B} Var\left(\tilde{X}_{i,j}^{Tot}\right)+\bar{\rho}^{Tot}\sum_{i,j\in B}\sum_{\substack{h,k\in B\\(h\neq i\,\vee\,k\neq j)}} SD\left(\tilde{X}_{i,j}^{Tot}\right)SD\left(\tilde{X}_{h,k}^{Tot}\right)\end{aligned} \tag{5.1}$$

The left side of Equation (5.1) refers to the variance of the one-year reserve, whereas the right side refers to the total run-off reserve (here superscripts "OY" and "Tot" indicate the one-year and the total run-off time horizon). The terms $\sum_{i,j\in B} Var\left(\tilde{X}_{i,j}^{OY}\right)$ and $\sum_{i,j\in B} Var\left(\tilde{X}_{i,j}^{Tot}\right)$ include the variance of each single cell of the lower triangle; the variances relative to the first diagonal (the forthcoming accounting year) match by construction in the total run-off and in the one-year evaluations, thus it can be neglected. We show by simulation that the variability of each single cell of the one-year reserve is lower than the one of the total run-off reserve. Therefore, the inequality of formula (5.1) is imputable to the covariance terms; hence, the average Pearson correlation coefficient of the one-year reserve exceeds the one related to a total run-off time horizon. The modeling explanation of this circumstance is ascribable to the use of two different approaches to assess the reserve under a total run-off and a one-year time horizon. When the whole lifetime of obligations is considered, we adopt the CRM, whereas when only the next 12 months are taken into account, the reserve is determined according to the re-reserving approach. The latter imposes the calculation, at each simulation step, of the first diagonal that it is adopted to estimate, in line with the underling deterministic model, the lower residual cells of the triangle. With regards to the Frequency-Severity method, at each iteration, according to the first simulated diagonal, the one-year approach calculates the Chain-Ladder development factors to estimate the remaining lower triangle. Therefore, by construction, the re-reserving approach induces a not negligible dependence between cells of the lower triangle; the average Pearson correlation coefficient affecting the triangle for the one-year reserve is higher than the one for the total run-off view, which, in our case study, is almost zero due to the perfect negative dependence between $\tilde{q}$

and $\tilde{p}$. The logical explanation related to a higher variability under a one-year time horizon evaluation, in respect of the total run-off, lies of course in the different time horizon taken into consideration. When we consider the whole lifetime of obligations, the random variables related to the cells of the lower triangle tend to compensate each other, especially thanks to the negative dependence of the structure variables, thus reducing the reserve variability. On the other hand, when the reserve is evaluated taking only the next 12 months into account, the random variables are not able to offset one another as much as they do over their whole lifetime, leading to a higher variability in respect of the one related to the total run-off time horizon. Indeed, the one-year evaluation of the reserve disregards the stochastic claims process over the next 12 months, not allowing the matching among the r.v.s $\tilde{X}_{i,j}$, driven by the negative dependence between $\tilde{q}$ and $\tilde{p}$, to get completely displayed. Moreover, under the Frequency-Severity method based on Chain-Ladder mechanics, the re-reserving approach stresses this particular case when the triangle is small due to the higher impact that the first simulated diagonal has on the residual lower triangle. In Table 15 we report the Pearson correlation coefficient estimated by simulation between the first diagonal of the lower triangle and the residual cells.

Table 15: Pearson correlation coefficient for both insurers between first diagonal and the remaining lower cells of triangle.

Dimension	**DELTA**		**OMEGA**	
	Tot. run-off	One-year	Tot. run-off	One-year
4x4	0.013	0.854	0.003	0.890
5x5	0.018	0.821	0.008	0.841
6x6	0.016	0.761	0.004	0.804
7x7	0.021	0.775	0.008	0.775
8x8	0.013	0.647	0.010	0.714
9x9	0.014	0.644	0.011	0.719
10x10	0.016	0.689	0.012	0.743
11x11	0.020	0.738	0.008	0.687
12x12	0.024	0.586	0.014	0.630

With regards to the different dimensions of the run-off triangle, Table 16 shows the SCR ratio and the ratio between the coefficient of variation of the reserve evaluated under a one-year time horizon and the one related to the total run-off reserve.

Table 16: Coefficient of variation of the one-year reserve in respect of the one related to the total run-off reserve and SCR ratio for both insurers.

Dimension	**DELTA**		**OMEGA**	
	CV_{OY}/CV_{Tot}	SCR ratio	CV_{OY}/CV_{Tot}	SCR ratio
4x4	112.01%	15.75%	113.89%	6.03%
5x5	108.96%	14.73%	113.78%	5.76%
6x6	108.79%	14.52%	111.85%	5.55%
7x7	103.15%	13.23%	110.38%	5.22%
8x8	107.10%	15.23%	117.32%	6.06%
9x9	104.05%	14.68%	110.79%	5.33%
10x10	97.30%	12.97%	103.66%	4.55%
11x11	92.69%	12.20%	105.71%	4.51%
12x12	100.55%	12.23%	112.54%	4.31%

As expected, the ratio between the coefficients of variation is higher for triangles with low dimension due to the higher correlation induced by the re-reserving approach. When the triangle dimension increases, the weight of the one-year coefficient of variation over the total run-off relative variability tends to decrease, but not necessarily in a monotonic way. Also the SCR ratio exhibits, in general, a decreasing trend in respect of the triangle dimension caused by the increasing number of reserved claims that highlights the diversification effect. If we compare the two insurers, the ratio between the coefficient of variation is higher for OMEGA, due to the greater impact that structure variables have on the total run-off reserve. On the other hand, the SCR ratio is lower for OMEGA because of the greater number of reserved claims that allow the insurer to diversify mainly the component of pooling risk.

In sum, the coefficient of variation of the one-year reserve in respect of the one assessed under a total run-off time horizon depends on three factors: the run-off triangle dimension, namely the impact that the future diagonal has on the remaining cells to be estimated, the level of dependence between structure variables and, in general, the characteristics of the data set. It is to be pointed out that from a mathematical point of view, it is not possible *a priori* to know the direction of the inequality between the coefficients of variation of the one-year and total run off reserve in respect of the value of correlation between $\tilde{q}$ and $\tilde{p}$.

6 Conclusions

As shown in [22] the estimation of structure variables embedded into the Collective Risk Model to stochastically evaluate the claims reserve is a key issue. In the present work, we developed a Bayesian approach to quantify the variability of structure variables implementing the Bayes' rule through uninformative prior distributions and data obtained by applying a bootstrapping-based procedure to run-off triangles integrated via Mack's formula. In addition, the dependence between structural risk factors has been investigated in a Bayesian manner: we proposed a joined resampling scheme aimed at capturing the inherent dependency of data. Through a case study we showed the impact that the dependence between structure variables has on claims reserve distribution, evaluated with respect to both the entire liability settlement period, the so-called total run-off approach, and the one-year time horizon, in order to assess the reserve risk capital requirement. When perfect negative linear dependence is addressed on structural risk factors, we come across a unique situation where the coefficient of variation of the one-year reserve exceeds the relative variability of the total run-off reserve. Starting from this circumstance, we analysed both the modeling connection and the logical link between the coefficient of variation of reserve appraised under the two time horizons.

In our opinion, the methodology developed in the present paper, making use of only historical data, allows us to estimate the magnitude and the dependence between structure variables both avoiding any sort of expert judgment and providing a coherent approach with the Collective Risk Model to assess structural risk factors. Moreover this approach fills a gap in existing research literature that is lacking methodologies designed to calibrate structure variables related to Collective Risk Theory models for reserve risk. Nevertheless, the use of Bayes' rule based on a selected parametric model turns to be a hard assumption to prove; further developments may consider likelihood-free frameworks, where the statistical model is defined in terms of a stochastic generating mechanism of data.

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Appendix A

Table 17: DELTA - Cumulative paid amounts (thousands of Euro).

DELTA												
AY/DY	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1993	38,364	76,319	91,669	97,769	100,947	103,649	105,152	106,513	107,521	108,419	108,707	110,502
1994	41,475	85,940	101,878	108,718	112,018	114,748	115,757	116,909	117,675	118,143	118,599	
1995	46,520	94,099	109,195	116,103	119,495	120,885	122,223	123,409	124,331	124,890		
1996	47,925	99,792	117,390	123,696	126,570	128,694	130,927	132,135	133,008			
1997	51,420	103,505	120,796	126,816	129,536	132,573	133,893	135,017				
1998	57,586	111,736	131,345	138,876	142,986	145,766	148,032					
1999	55,930	110,872	131,819	142,318	148,182	151,495						
2000	51,005	104,197	126,016	134,380	139,095							
2001	51,693	103,266	121,933	130,767								
2002	54,954	106,565	125,169									
2003	59,763	113,506										
2004	60,361											

Table 18: DELTA - Cumulative number of paid claims.

DELTA												
AY/DY	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1993	34,433	48,229	49,819	50,387	50,665	50,816	50,920	50,975	51,004	51,036	51,053	51,108
1994	35,475	49,193	50,693	51,241	51,451	51,584	51,635	51,679	51,706	51,726	51,742	
1995	37,004	50,824	52,351	52,788	52,981	53,053	53,099	53,128	53,145	53,162		
1996	37,038	50,669	52,131	52,631	52,795	52,875	52,938	52,977	53,003			
1997	36,849	50,265	51,828	52,251	52,432	52,539	52,620	52,662				
1998	39,171	51,772	53,364	53,924	54,197	54,372	54,527					
1999	37,492	49,774	51,831	52,571	52,961	53,248						
2000	34,188	46,433	48,371	49,132	49,427							
2001	31,308	42,051	43,958	44,597								
2002	30,357	40,474	42,085									
2003	30,717	41,799										
2004	30,590											

Table 19: OMEGA - Cumulative paid amounts (thousands of Euro).

OMEGA

AY/DY	1	2	3	4	5	6	7	8	9	10	11	12
1993	193,474	366,091	453,292	499,090	528,858	548,653	568,435	585,750	599,122	611,674	620,504	648,446
1994	199,854	368,820	449,363	490,019	519,072	540,193	560,158	574,406	585,126	598,810	604,818	
1995	225,578	412,343	505,692	553,301	584,272	610,562	628,183	646,594	661,255	668,846		
1996	256,398	493,076	598,692	649,864	687,202	711,287	732,041	744,122	758,260			
1997	282,956	546,152	666,535	730,224	767,444	796,684	819,804	835,313				
1998	292,428	576,829	718,229	774,619	814,814	842,770	872,756					
1999	312,350	597,857	729,544	804,796	851,345	890,076						
2000	327,673	635,665	797,182	875,147	927,842							
2001	339,899	666,179	852,090	953,363								
2002	371,275	757,122	950,128									
2003	388,025	778,762										
2004	398,686											

Table 20: OMEGA - Cumulative number of paid claims.

OMEGA

AY/DY	1	2	3	4	5	6	7	8	9	10	11	12
1993	284,236	395,010	407,622	412,348	414,554	415,754	416,509	417,012	417,415	417,820	418,070	418,767
1994	274,524	367,933	379,865	383,968	385,970	387,096	387,811	388,278	388,747	389,065	389,285	
1995	284,017	374,422	386,345	390,312	392,114	393,119	393,780	394,370	394,770	395,011		
1996	299,605	400,289	412,734	416,404	417,964	418,849	419,473	419,855	420,157			
1997	308,092	411,618	426,850	431,376	433,420	434,608	435,384	435,926				
1998	295,813	391,103	405,954	410,705	413,070	414,337	415,277					
1999	288,418	381,906	399,078	405,364	408,304	410,063						
2000	285,940	387,422	407,558	415,654	419,682							
2001	290,023	386,767	409,717	418,962								
2002	280,008	374,305	396,095									
2003	277,412	369,192										
2004	252,239											

Appendix B

Table 21: DELTA - Uninformative priors: expected values of posterior distributions.

DELTA		
Type of prior	**Uniform**	**Jeffreys**
$E\left(\tilde{A}_{\tilde{q}} \mid \tilde{\sigma}_{\tilde{q}}\right)$	64.726	48.576
$E\left(\tilde{B}_{\tilde{q}} \mid \tilde{\sigma}_{\tilde{q}}\right)$	3,008.640	2,254.114
$E\left(\tilde{A}_{\tilde{p}} \mid \tilde{\sigma}_{\tilde{p}}\right)$	32.160	24.164
$E\left(\tilde{B}_{\tilde{p}} \mid \tilde{\sigma}_{\tilde{p}}\right)$	1,006.136	798.222

Table 22: OMEGA - Uninformative priors: expected values of posterior distributions.

OMEGA		
Type of prior	**Uniform**	**Jeffreys**
$E\left(\tilde{A}_{\tilde{q}} \mid \tilde{\sigma}_{\tilde{q}}\right)$	17.305	13.039
$E\left(\tilde{B}_{\tilde{q}} \mid \tilde{\sigma}_{\tilde{q}}\right)$	623.993	467.282
$E\left(\tilde{A}_{\tilde{p}} \mid \tilde{\sigma}_{\tilde{p}}\right)$	123.468	92.717
$E\left(\tilde{B}_{\tilde{p}} \mid \tilde{\sigma}_{\tilde{p}}\right)$	4,597.385	3,449.045

Appendix C

Let $\tilde{X}$ be a generic random variable following a gamma distribution with parameters $\alpha > 0$ and $\beta > 0$. The density function is:

$$f_{\tilde{x}}(X) = \frac{\beta^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x}, \qquad x \in \mathbb{R}^{+},$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt$ is the gamma function. The j-th moment about zero is

$$E\left(\tilde{X}^{j}\right) = \frac{\alpha(\alpha+1)\ldots(\alpha+j-1)}{\beta^{j}},$$

Whereas the j-th cumulant is

$$K_j\left(\tilde{X}\right) = \frac{(j-1)!\alpha}{\beta^{j}}.$$

We here compute the characteristics of the structure variable $\tilde{q}$ (similar consideration hold for $\tilde{p}$). The structure variable related to the claim count follows a gamma distribution with same parameters:

$$\tilde{q} \sim Gamma\left(\frac{1}{\tilde{\sigma}_{\tilde{q}}^{2}}; \frac{1}{\tilde{\sigma}_{\tilde{q}}^{2}}\right),$$

where $\tilde{\sigma}_{\tilde{q}}^{2}$ is itself a random variable. In general it is possible to compute the moments of the structure variable $\tilde{q}$ without knowing the distributional form of the r.v. $\tilde{\sigma}_{\tilde{q}}$. The expected value is given by

$$E(\tilde{q}) = E_{\tilde{\sigma}_{\tilde{q}}}\left[E_{\tilde{q}}\left(\tilde{q} \mid \tilde{\sigma}_{\tilde{q}}\right)\right] = E_{\tilde{\sigma}_{\tilde{q}}}\left[\frac{\tilde{\sigma}_{\tilde{q}}^{2}}{\tilde{\sigma}_{\tilde{q}}^{2}}\right] = 1,$$

therefore, the mean of the structure variable remains equal to 1. With regard to the variance, it can be compute as:

$$\begin{aligned} Var(\tilde{q}) &= E_{\tilde{\sigma}_{\tilde{q}}}\left[Var_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{q} \mid \tilde{\sigma}_{\tilde{q}}\right)\right] + Var_{\tilde{\sigma}_{\tilde{q}}}\left[E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{q} \mid \tilde{\sigma}_{\tilde{q}}\right)\right] \\ &= E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{\sigma}_{\tilde{q}}^{2}\right) + Var_{\tilde{\sigma}_{\tilde{q}}}(1) = E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{\sigma}_{\tilde{q}}^{2}\right), \end{aligned}$$

showing that the second raw moment of the r.v. $\tilde{\sigma}_{\tilde{q}}$ represents the variance of $\tilde{q}$. The relative variability is equal to the square root of the variance:

$$CV(\tilde{q}) = \sqrt{E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{\sigma}_{\tilde{q}}^{2}\right)}.$$

Finally, the skewness equals:

$$\gamma(\tilde{q}) = \frac{\mu_3(\tilde{q})}{\left[Var(\tilde{q})\right]^{\frac{3}{2}}} = \frac{E\left(\tilde{q}^3\right) - 3E(\tilde{q})Var(\tilde{q}) - E(\tilde{q})^3}{\left[Var(\tilde{q})\right]^{\frac{3}{2}}}$$

$$= \frac{1 + 3E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{\sigma}_{\tilde{q}}^2\right) + 2E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{\sigma}_{\tilde{q}}^4\right) - 3E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{\sigma}_{\tilde{q}}^2\right) - 1}{\left[Var(\tilde{q})\right]^{\frac{3}{2}}} = \frac{2E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{\sigma}_{\tilde{q}}^4\right)}{\left[Var(\tilde{q})\right]^{\frac{3}{2}}},$$

where it hold that $E\left(\tilde{q}^3\right) = E_{\tilde{\sigma}_{\tilde{q}}}\left[E_{\tilde{\sigma}_{\tilde{q}}}\left(\tilde{q}^3 \mid \tilde{\sigma}_{\tilde{q}}\right)\right] = E_{\tilde{\sigma}_{\tilde{q}}}\left(1 + 3\tilde{\sigma}_{\tilde{q}}^2 + 2\tilde{\sigma}_{\tilde{q}}^4\right)$. It is to be noted that the third cumulant of $\tilde{q}$, i.e. the numerator of skewness, depends only on the 4-th central moment of $\tilde{\sigma}_{\tilde{q}}$. Under the assumption that the r.v. $\tilde{\sigma}_{\tilde{q}}$ follows a gamma distribution of parameters α and β, $\tilde{\sigma}_{\tilde{q}} \sim Gamma(\alpha;\beta)$, the characteristics of the structure variable can be rewritten as follow.

Variance:

$$Var(\tilde{q}) = \frac{\alpha(\alpha+1)}{\beta^2}.$$

Coefficient of variation:

$$CV(\tilde{q}) = \frac{\sqrt{\alpha(\alpha+1)}}{\beta}.$$

Skewness:

$$\gamma(\tilde{q}) = \frac{2(\alpha+2)(\alpha+3)}{\beta\sqrt{\alpha(\alpha+1)}}.$$

Appendix D

Table 23: Uninformative priors: expected values of posterior distribution for both insurers.

Posterior expected values			
Insurer	**Type of prior**	**Uniform**	**Jeffreys**
DELTA	$E\left(\tilde{C};\tilde{\rho}\right)$	2.272	1.752
	$E\left(\tilde{D};\tilde{\rho}\right)$	14.049	10.411
OMEGA	$E\left(\tilde{C};\tilde{\rho}\right)$	4.031	3.102
	$E\left(\tilde{D};\tilde{\rho}\right)$	12.987	9.779