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Discontinuous Transformations and Lorenz Curves

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Abstract

In this study we reconsider the effect of variable transformations on the redistribution of income. Under the assumption that the theorems should hold for all income distributions, earlier given conditions are both necessary and sufficient. Different versions of the conditions are compared. We also consider the consequences if we drop the explicit continuity restriction on the transformations. One main result is that continuity is a necessary condition if one pursues that the income inequality should remain or be reduced. In our earlier studies concerning tax policies the assumption that differentiable transformations satisfying a derivative condition could be reduced to transformations satisfying only a continuity condition.

Mathematics Subject Classification: D63, H20, H21
Keywords: Lorenz dominance, Lorenz ordering, Post-tax distribution

1 Introduction

It is a well-known fact that variable transformations are valuable when one studies the effect of tax and transfer policies on the income inequality. Usually the transformation is assumed to be positive, monotone increasing and continuous. Under the assumption that the theorems should hold for all income distributions, earlier given conditions are both necessary and sufficient [1] and we reconsider the effect of variable transformations on the redistribution of income. Different versions of the conditions are compared ([1] - [4]). We drop the explicit

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assumption of continuity of the transformations, but it can be implicitly included in the necessary and sufficient conditions. One main result is that continuity is a necessary condition if one pursues that the income inequality should remain or be reduced. In addition, in our earlier studies of classes of tax policies, the results were based on the assumption that the transformations were differentiable satisfying a derivative condition [5], [6]. It is possible to reduce this assumption to a continuity condition. is the text of the introduction.

2 Main Results

2.1 Properties of Continuous Transformations

Consider the income *X* with the distribution function $F_X(x)$, the mean μ_X, and the Lorenz curve $L_X(p)$. We assume that *X* is defined for $x \geq 0$. If we assume that the density function $f_X(x)$ exists, we obtain the formulae

$$p = F(x_p), \tag{1}$$

$$\mu_X = \int_0^\infty x f_X(x) dx \tag{2}$$

and

$$L_X(p) = \frac{1}{\mu_X} \int_0^{x_p} x f_X(x) dx. \tag{3}$$

We consider the transformation $Y = u(X)$, where $u(\cdot)$ is non-negative and monotone increasing. The transformation can be considered as a tax or a transfer policy and consequently, the transformed variable is the post-tax or post-transfer income, respectively. For the transformed variable *Y* we obtain the distribution function

$$F_Y(y) = P(Y \leq y) = P(u(X) \leq y) = P\left(X \leq u^{-1}(y)\right) = F_X\left(u^{-1}(y)\right)$$

Using this result we obtain the mean and the Lorenz curve for the variable *Y*. They are

$$\mu_Y = \int_0^\infty u(x) f_X(x) dx \tag{4}$$

and

$$L_Y(p) = \frac{1}{\mu_Y} \int_0^{x_p} u(x) f_X(x) dx . \quad (5)$$

A fundamental theorem concerning the effect of income transformations on Lorenz curves and Lorenz dominance is

Theorem 1. ([1], [2], [3]). *Let X be an arbitrary non-negative, random variable with the distribution* $F_X(x)$, *mean* μ_X *and Lorenz curve* $L_X(p)$. *Let* $u(x)$ *be non-negative, continuous and monotone-increasing and let* $\mu_Y = E(u(X))$ *exist. Then the Lorenz curve* $L_Y(p)$ *of* $Y = u(X)$ *exists and the following results hold*

(i) $L_Y(p) \ge L_X(p)$ *if* $\frac{u(x)}{x}$ *is monotone decreasing,*

(ii) $L_Y(p) = L_X(p)$ *if* $\frac{u(x)}{x}$ *is constant and*

(iii) $L_Y(p) \le L_X(p)$ *if* $\frac{u(x)}{x}$ *is monotone increasing.*

According to this theorem we obtain in (i) a sufficient condition that the transformation $u(x)$ generates a new income distribution which Lorenz dominates the initial one ($L_i(p)$ in Figure 1)

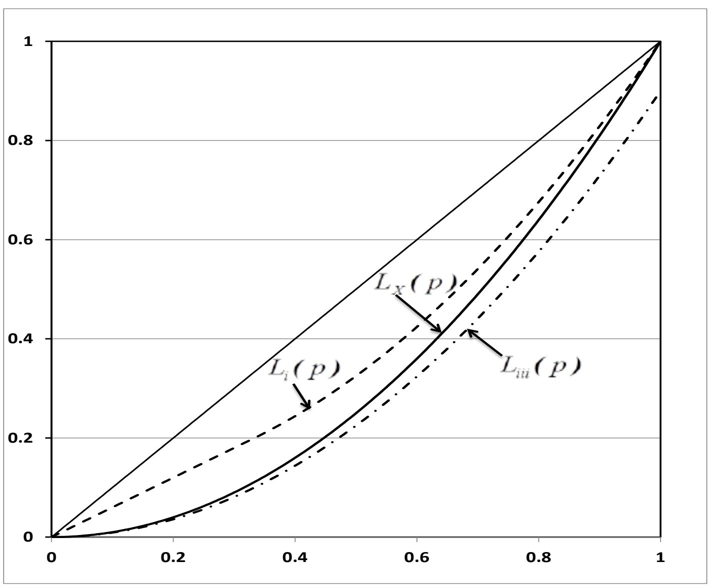

Figure 1: Lorenz Curves and Lorenz Dominance

If we analyse the proof of the case (i) in Theorem 1 in [2] we observe that the difference $L_Y(p) - L_X(p)$ can be written

$$D(p) = L_Y(p) - L_X(p) = \int_0^{x_p} \left(\frac{u(x)}{\mu_Y} - \frac{x}{\mu_X} \right) f(x)\,dx = \int_0^{x_p} \left(\frac{u(x)}{x} - \frac{\mu_Y}{\mu_X} \right) \frac{x}{\mu_Y} f(x)\,dx. \quad (6)$$

In any case, $D(0) = D(1) = 0$. In order to obtain Lorenz dominance the difference $D(p)$ must start from zero and then attain positive values and after that decrease back to zero and the integrand in (6) must start from positive (non-negative) values and then change its sign and become negative. Consequently, $\frac{u(x)}{x}$ has to be a decreasing function. The condition is necessary if the rule should hold for all income distributions $F_X(x)$ [1]. Assume that the quotient $\frac{u(x)}{x}$ is both increasing and decreasing. If $\frac{u(x)}{x}$ is monotonously increasing for all $x > 0$ then the proposition (iii) holds and this case can be ignored. Let a transformation $u(x)$ satisfy the initial conditions (non-negative, continuous and monotone increasing) and let $\frac{u(x)}{x}$ be increasing within some interval ($0 < a < x < b < \infty$). Now we present a distribution such that the transformed variable $Y = u(X)$ does not Lorenz dominate the initial variable *X*. Consider a distribution with a continuous density function,

$$f_X(x) = \begin{cases} 0 & 0 \le x < a \\ f_0(x) > 0 & a \le x \le b \\ 0 & x > b \end{cases} \quad (7)$$

For the pair ($f_X(x), u(x)$) the formula (6) can be written

$$D(p) = \int_a^{x_p} \frac{x}{\mu_Y} \left(\frac{u(x)}{x} - \frac{\mu_Y}{\mu_X} \right) f_X(x)dx, \quad (8)$$

where $a \le x_p \le b$.

We observe that $D(0) = D(1) = 0$, that Theorem 1(iii) holds and that the transformation results in a new variable *Y* which is Lorenz dominated by the initial variable *X*. Hence, if we demand that the transformed variable $Y = u(X)$ shall Lorenz dominate *X* for all distributions $F_X(x)$, then the condition in Theorem 1(i) is necessary ([1], [7], Chapter 8).

Hemming and Keen [4] gave an alternative condition for the Lorenz dominance. Their condition is, with our notations, that for a given distribution $F_X(x)$ the function $u(x)$ crosses the line $\frac{\mu_Y}{\mu_X} x$ once from above, that is that

$\frac{u(x)}{x}$ crosses the level $\frac{\mu_Y}{\mu_X}$ once from above. We observe that if their condition holds then the integrand in (6) starts from positive values, changes its sign once and ends up with negative values and their condition is equivalent with our condition. For the example considered above, the Hemming-Keen condition is not satisfied. The integrand is zero for $x < a$ and for $x > b$. For $a \le x \le b$ the ratio $\frac{u(x)}{x}$ is increasing and if it crosses $\frac{\mu_Y}{\mu_x}$ it cannot do it from above. Consequently if $\frac{u(x)}{x}$ is not monotone decreasing then there are distributions for which the Hemming-Keen condition does not hold.

On the other hand if we assume that $\frac{u(x)}{x}$ is monotone decreasing then $\frac{u(x)}{x}$ satisfies the condition "crossing once from above for every distribution $F_X(x)$". Hence, our condition and Hemming-Keen condition are also equivalent as necessary conditions. In a similar way we can prove that if the other results in Theorem 1 should hold for every income distribution the conditions in (ii) and in (iii) are also necessary.

2.2 The Effect of Discontinuities in the Transformation *u*(*x*)

The results obtained, indicate that if $\frac{u(x)}{x}$ is continuous and monotone increasing even in a short interval, then there are income distributions such that the transformation $u(x)$ cannot result in Lorenz dominance. What can be said if $u(x)$ is discontinuous? Assume that $u(x)$ is still positive and monotone increasing. Assume furthermore, that $E(u(X)) = \mu_Y$ exists for every stochastic variable *X* with finite mean μ_X. The discontinuities of $u(x)$ can only consist of finite positive jumps. For realistic models within this framework, the number of jumps can be assumed to be finite or countable. Assume that elsewhere $u(x)$ satisfies all the other conditions including the condition in Theorem 1(i). We will prove that if $u(x)$ is discontinuous there exists a distribution $F_X(x)$ such that the transformation $Y = u(X)$ does not Lorenz dominate the initial variable *X*. Again we follow the arguments given by Jakobsson [1]. However, the discontinuity demands a more detailed reasoning. A detailed presentation of the discontinuity case was earlier given in [8].

Let $a>0$ be a discontinuity point, such that $\lim_{x\to a-} u(x)=u_0$ and $\lim_{x\to a+} u(x)=u_0+d$, where the jump $d>0$. (The notation $\lim_{x\to a-} u(x)$ indicates limit from the left and $\lim_{x\to a+} u(x)$ limit from the right.) We do not assume anything about how $u(x)$ is defined in the point a. Choose $h>0$ so small that the point a is the only discontinuity point within the interval $(a-h,a+h)$. (Later we may reduce the interval even more). Let t and z be arbitrary values satisfying the inequalities

$$a-h<t\le a\le z<a+h.$$

If $u(x)$ is monotone increasing we have $u(t)\le u_0<u_0+d\le u(z)$ and

$$lim_{t\to a-}\left(\frac{u(t)}{t}\right)=\frac{u_0}{a}<\frac{u_0+d}{a}=lim_{z\to a+}\left(\frac{u(z)}{z}\right).$$

Hence, the quotient $\frac{u(x)}{x}$ cannot be monotone decreasing within the interval $(a-h,a+h)$. Consider a variable *X*, having the symmetric density function

$$f_X(x)=\begin{cases}0 & x<a-h\\ \frac{1}{h}\left(1-\frac{1}{h}|a-x|\right) & a-h\le x\le a+h\,.\\ 0 & x>a-h\end{cases} \tag{9}$$

The mean $E(X)=\mu_X=a$. For the transformed variable $Y=u(X)$ the mean is

$$\mu_Y=E(Y)=\int_{a-h}^{a}u(x)f_X(x)dx+\int_{a}^{a+h}u(x)f_X(x)dx= \tag{10}$$

$$=u(\alpha_1)\int_{a-h}^{a}f_X(x)dx+u(\alpha_2)\int_{a}^{a+h}f_X(x)dx=\frac{1}{2}\big(u(\alpha_1)+u(\alpha_2)\big),$$

where $a-h<\alpha_1<a$ and $a<\alpha_2<a+h$. If $h\to 0$ then $\mu_Y\to u_0+\frac{1}{2}d$.

Assume furthermore, that we have chosen *h* so small that $\mu_Y>u_0+\frac{1}{4}d$. Consider now

$$D(p)=L_Y(p)-L_X(p)=\int_{a-h}^{x_p}\frac{x}{\mu_Y}\left(\frac{u(x)}{x}-\frac{\mu_Y}{\mu_X}\right)f_X(x)dx\,, \tag{11}$$

where $F_X(x_p)=p$. In order to obtain Lorenz dominance the integrand must start from positive (non-negative) values and then change its sign and become negative in such a manner that the difference $D(p)$ starts from zero and then attains positive values and after that it decreases back to zero. Within the interval

$\left(a-h, a+h\right)$ the sign of the integrand depends on the factor $\frac{u(x)}{x}-\frac{\mu_Y}{\mu_X}$, which starts from the value

$$\frac{u(a-h)}{a-h}-\frac{\mu_Y}{a}\le\frac{u_0}{a-h}-\frac{u_0+\frac{1}{4}d}{a}\le\frac{-\frac{1}{4}ad+h(u_0+\frac{1}{4}d)}{a(a-h)}.$$

If we assume that h satisfies the earlier conditions and in addition, the condition $h<\frac{ad}{4u_0+d}$, the parenthesis in (11) starts from negative values and consequently, the whole integrand is negative and $D(p)$ starts from negative values. For the corresponding income distribution the transformed variable Y does not Lorenz dominate the initial variable X. Hence, the continuity of $u(x)$ is a necessary condition if we demand that the transformed variable should Lorenz dominate the initial variable for every distribution. From this it follows that if the condition in Theorem 1(i) has to be necessary it implies continuity and hence, an explicit statement of continuity can be dropped. If we study the condition in (ii) we observe that $u(x)=kx$ and consequently, $u(x)$ has to be continuous.

However, in the case (iii) the discontinuity does not jeopardize the monotone increasing property of the quotient $\frac{u(x)}{x}$ and the result in Theorem 1 (iii) holds even if the function is discontinuous. Therefore, also in this case we can drop the explicit continuity assumption.

Summing up, for arbitrary distributions, $F_X(x)$, the conditions (i), (ii), and (iii) in Theorem 1 are both necessary and sufficient for the dominance relations and an additional assumption about the continuity of the transformation $u(x)$ can be dropped. We obtain the more general theorem.

Theorem 2. ([8], [9], [10]) *Let X be an arbitrary non-negative, random variable with the distribution $F_X(x)$, mean μ_X and the Lorenz curve $L_X(p)$, let $u(x)$ be a non-negative, monotone increasing function and let $Y=u(X)$ and $E(Y)=\mu_Y$ exist. Then the Lorenz curve $L_Y(p)$ of Y exists and the following results hold:*

(i) $L_Y(p)\ge L_X(p)$ *if and only if* $\frac{u(x)}{x}$ *is monotone-decreasing*

(ii) $L_Y(p)=L_X(p)$ *if and only if* $\frac{u(x)}{x}$ *is constant*

(iii) $L_Y(p)\le L_X(p)$ *if and only if* $\frac{u(x)}{x}$ *is monotone-increasing.*

Remark. From the discussion above it follows that only in the case (iii) the transformation $u(x)$ can be discontinuous.

Now, we analyse the effect of a finite step in $u(x)$ on the Lorenz curve. We use the notations presented above. Let $t \le a \le z$, $r = F_X(t)$, $q = F_X(a)$ and $s = F_X(z)$. Consider the difference

$$\Delta L_Y = L_Y(F_X(z)) - L_Y(F_X(t)) = \frac{1}{\mu_Y}\int_r^s u(F_X^{-1}(p))dp =$$

$$= \frac{1}{\mu_Y}\int_r^q u(F_X^{-1}(p))dp + \frac{1}{\mu_Y}\int_q^s u(F_X^{-1}(p))dp = \frac{u(\alpha_1)}{\mu_Y}(q-r) + \frac{u(\beta_1)}{\mu_Y}(s-q),$$

where $t \le \alpha_1 \le a$ and $a \le \beta_1 \le z$.

When $t \to a-$ and $z \to a+$, then $q - r \to 0$, $s - q \to 0$ and $\Delta L_Y \to 0$. Hence, although the transformation $u(x)$ is discontinuous in the point *a*, the Lorenz curve is continuous. However, it is not differentiable. For every $t < a$ we obtain

$$\Delta L_Y = L_Y(q) - L_Y(r) = \frac{1}{\mu_Y}\int_r^q u\left(F_X^{-1}(p)\right)dp = \frac{u(\eta)}{\mu_Y}(q-r)$$

where $t < \eta < a$. We obtain $\frac{\Delta L_Y}{q-r} = \frac{u(\eta)}{\mu_Y}$. When $q - r \to 0+$ then $\eta \to a-$ and $\frac{\Delta L_Y}{q-r} \to \frac{u_0}{\mu_Y}$. Hence, $L_Y(p)$ has the left derivative $\left(\frac{dL_Y(p)}{dp}\right)_{p=q-} = \frac{u_0}{\mu_Y}$.

For every $z > a$ we obtain

$$\Delta L_Y = L_Y(s) - L_Y(q) = \frac{1}{\mu_Y}\int_q^s u(F_X^{-1}(p))dp = \frac{u(\varsigma)}{\mu_X}(s-q),$$

where $a < \varsigma < z$. We obtain

$$\frac{\Delta L_Y}{s-q} = \frac{u(\varsigma)}{\mu_Y}.$$

When $s - q \to 0+$ then $\eta \to a+$ and $\frac{\Delta L_Y}{s-q} \to \frac{u_0 + d}{\mu_Y}$.

Hence, $L_Y(p)$ has the right derivative

$$\left(\frac{dL_Y(p)}{dp}\right)_{p=q+} = \frac{u_0 + d}{\mu_Y} \ne \frac{u_0}{\mu_Y} = \left(\frac{dL_Y(p)}{dp}\right)_{p=q-}. \tag{12}$$

Consequently, $L_Y(p)$ is continuous in the point $q = F_X(a)$ but it is not differentiable and has a cusp for $p = q$.

Remark. If the transformation $u(x)$ is continuous then $d=0$ and we obtain equality in (12) and the Lorenz curve is differentiable with the derivative $L_Y'(p)=\frac{y_p}{\mu_Y}$.

In our earlier studies we noted that realistic tax policies demand continuous transformations. In addition the results concerning tax policies were based on the assumption of differentiable transformations satisfying a derivative condition ([5], [6]). Fellman ([10]) stressed that this restriction could be reduced to non-differentiable transformations satisfying a continuity condition. Consequently, the results in this study can only be applied on transfer policies.

3 Discussion

In this study we reconsidered the effect of variable transformations on the redistribution of income. The aim was to generalise the conditions considered in earlier papers. Particularly we were interested if we can drop the assumptions of continuity of the transformations. We have obtained that, if we demand sufficient and necessary conditions, theorems earlier obtained, still hold and the continuity assumption can be included in the general conditions. The main result is that continuity is a necessary condition if one pursues that the income inequality should remain or be reduced.

Studies of the class of tax policies indicated that the differentiability, earlier assumed, can be dropped but if one wants to retain the realism of the class the transformations should still be continuous and satisfy the restriction $\Delta\overline{u}(x)\leq\Delta x$. The earlier results obtained and presented in [5] and [6] still hold.

Empirical applications of the optimal policies of a class of transfer policies and the class of tax policies considered here have been discussed in Fellman et al. ([11], [12]). There we developed "optimal yardsticks" to gauge the effectiveness of given real tax and transfer policies in reducing inequality.

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References

[1] U. Jakobsson, On the measurement of the degree of progression, *J. Publ. Economics*, **5**, (1976), 161-168.

[2] J. Fellman, The effect of transformations on Lorenz curves, *Econometrica*, **44**, (1976), 823-824.
[3] N.C. Kakwani, Applications of Lorenz curves in economic analysis. *Econometrica*, **45**, (1977), 719-727.
[4] R. Hemming and M.J. Keen, Single crossing conditions in comparisons of tax progressivity. *J. Publ. Economics*, **20**, (1983), 373-380.
[5] J. Fellman, Mathematical properties of classes of income redistributive policies. *European Journal of Political Economy*, **17**, (2001), 195-209.
[6] J. Fellman, The redistributive effect of tax policies, *Sankhya*, *Ser. B.*, **64**, (2002), 1-11.
[7] P.J. Lambert, *The Distribution and Redistribution of Income: A mathematical analysis*, (3rd edition), Manchester, Manchester University Press, xiv+313 pp., 2001.
[8] J. Fellman, Discontinuous Transformations, Lorenz Curves and Transfer Policies, *Social Choice and Welfare*, **33**, (2009), 335–342, doi: 10.1007/s00355-008-0362-4.
[9] J. Fellman, Properties of Lorenz curves for transformed income distributions, *Theoretical Economics Letters*, **2**(5), (2012), 487-493, doi:10.4236/tel.2012.25091, http://www.SciRP.org/journal/tel
[10] J. Fellman, Properties of Non-Differentiable Tax Policies, *Theoretical Economics Letters*, **3**(3), (2013), 142-145, doi:10.4236/tel.2013.33022, http://www.scirp.org/journal/tel
[12] J. Fellman, M. Jäntti and P.J. Lambert, Optimal tax-transfer systems and redistributive policy: the Finnish experiment, Swedish School of Economics and Business Administration, *Working Paper*, **324**, (1996), 16 pp.
[13] J. Fellman, M. Jäntti and P.J. Lambert, Optimal tax-transfer systems and redistributive policy, *Scandinavian Journal of Economics*, **101**, (1999), 115-126.