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Chaos Detection in Economic Time Series: Metric versus Topological Tools

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Abstract

From an empirical point of view, it is difficult to distinguish between fluctuations provoked by random shocks and endogenous fluctuations determined by the nonlinear nature of the relation between economic aggregates. For this purpose, chaos tests are developed to investigate the chaotic phenomena of basic features: nonlinearity, fractal attractor, and sensitivity to initial conditions. The application of these tests to economic and financial time series produced controversial results. Investigators found substantial evidence for nonlinearity but relatively weak evidence for chaos per se.
The aim of the paper is twofold. In the first place, to compare the different techniques with which to analyse chaotic time series highlighting their potentiality and limitations. Secondly, to apply in an empirical exercise a topological tool - Recurrence Analysis - using data already analysed with metric tests in order to show whether the result of this analysis could change if performed with tools more appropriate for discovering chaos in short and noisy time series.

JEL classification numbers: C4, C49
Keywords: Economic time series, chaos tests, metric and topological tools

1 Introduction

The way to identify movements of economic indicators is a central and conflicting debate in macroeconomics that arises around two opposite approaches:
The exogenous-shocks-equilibrium and the endogenous-cycles-disequilibrium. For the former, fluctuations are deviations from a steady growth path determined by exogenous "shocks" such as fiscal and monetary policy changes, and changes in technology. Stochastic exogenous disturbances are superimposed upon (usually linear) deterministic models to produce the stochastic appearance of actual economic time series [1].

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Economic fluctuations, seemingly as complex as noise, are considered fluctuations like identically independently distributed (i.i.d.) events [2].
According to the latter, deviations from growth trends are consequences of endogenous shocks, which arise from imperfections of the market. In this sense endogenous cycles are represented by deterministic oscillators including harmonic cycle and limit cycle [3].
But the real economic time-series do not show the kind of regularity and symmetry that is predicted by those models. Irregular frequencies and different amplitudes are the real feature of fluctuations in economic indicators.
The poor results in terms of description and forecasting obtained by using those models and the contrast with real behaviour of economic variables gave place to a challenging focus that considers economic fluctuations responding to a non-linear structure.
The nonlinear approach is well suited for examining economic fluctuations because it is able to capture stylized facts observed in many financial and economic time series such as asymmetries, jumps and time irreversibility, and to model any oscillating phenomenon.
In the literature, there many examples of nonlinear methods used to analyse time series. These are the so-called "ARCH-type" models proposed by [4] and generalised by [5]. Among the "ARCH-type" models - Exponential GARCH, Asymmetric Power ARCH, Threshold GARCH - the so-called integrated model (IGARCH) and fractionally integrated model (FIGARCH) have recently been the most popular. These models are based on the assumption that data are nonlinear stochastic functions of their past values. However, it is also possible that data can be generated by deterministic processes.
Nonlinear deterministic systems with a few degrees of freedom can create output signals that appear complex and mimic stochastic signals from the point of view of conventional time series analysis but are chaotic.
Economists began to look at chaos theory in the late 1980s with important works like those by [6, 7, 8, 9, 10], just to name few. A common feature of these chaos models is that nonlinear dynamics tend to arise as the result of relaxing the assumptions underlying the competitive market general equilibrium approach.
However, showing that a mathematical model exhibits chaotic behaviour is no proof that chaos is also present in the corresponding experimental system. To convincingly show that an experimental system behaves chaotically, chaos has to be directly identified from the experimental data [11].
In economics data sets are the outcome of a complex process including institutional or structural changes and monetary regime switches, shocks, wars, political crises etc. The rich nature as well as the impact of those changes reveals interesting features in time series (structural instability and nonlinearity) that needs to be studied by developing new techniques, able to filter these complex dynamics [12]. The application of nonlinear tools for identifying causal relationships between economic variables can provide information that the linear tools could miss [13].
Researchers in economics and finance have been interested in testing nonlinear dependence and chaos for almost two decades. A wide variety of reasons for this interest have been suggested, including an attempt to improve the forecasting accuracy of linear time series models and to better explain the dynamics of the underlying variables of interest using a richer class of models than that permitted by limiting the set to the linear case.
During these two decades the search for chaos in economics has gradually became less enthusiastic, as no empirical support for the presence of chaotic behaviours in economics has been found. The literature did not provide a solid support for chaos as a consequence

of the high noise level that exists in most economic time series, the relatively small sample sizes of data, and the weak robustness of chaos tests for these data.

From an empirical point of view, it is difficult to distinguish between fluctuations provoked by random shocks and endogenous fluctuations determined by the nonlinear nature of the relation between economic aggregates. For this purpose, chaos tests are developed to investigate the chaotic phenomena of basic features: nonlinearity, fractal attractor, and sensitivity to initial conditions. The application of these tests to economic and financial time series produced controversial results. Investigators found substantial evidence for nonlinearity but relatively weak evidence for chaos per se.

Starting from this state of the art, the aim of the paper is twofold. In the first place, to compare the different techniques with which to analyse chaotic time series highlighting their potentiality and limitations. Secondly, to apply in an empirical exercise a topological tool - Recurrence Analysis - using data already analysed with metric tests in order to show whether the result of this analysis could change if performed with tools more appropriate for discovering chaos in short and noisy time series.

The paper is set up as follows. In section 2 metric and dynamical tools features are described. Section 3 focuses on the description of the alternative to overcome the weakness of these tests. In section 4 we describe the application of Recurrence Analysis with Visual Recurrence Analysis software to macroeconomic time series analysed by [14] in their paper "International chaos?" using metric and dynamical tools. Section 5 presents the main results and conclusions.

2 Metric and Dynamical Tools

There are different signs, or invariants, that are representative of chaos in a system: nonlinearity, dependence on initial conditions, and presence of an attractor with fractal dimension, known as a strange attractor. Based on these signs, several tools have been developed in order to investigate the chaotic properties of a system from a time series. These tools could be classified in metric - correlation dimension[2] and BDS test, dynamical- Lyapunov exponent[3], and topological [15] - Close Return Test and Recurrence Analysis.

2.1 Correlation Dimension.

A necessary but not sufficient condition in order to define a system as being chaotic is that the strange attractor has a fractal dimension.

The notion of dimension refers to the degree of complexity of a system expressed by the minimum number of variables that is needed to replicate the system [16]. For example, a cube has three dimensions, a square has two dimensions, and a line has one. A chaotic system has non-integer dimensionality called fractal dimension.

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[2]The correlation dimension is a metric method because it is based on the computation of distances on the system's attractor.

[3]Lyapunov exponent instead, is a dynamical method because it is based on computing the way close to where orbits diverge.

In the literature, there are many methods[4] for calculating the fractal dimension (Hausdorff dimension, the box-counting dimension, the information dimension, and the correlation dimension), which nevertheless do not provide equivalent measures [17]. Among these different algorithms, the correlation dimension proposed by [18], based on phase space reconstructions of the process to estimates[5], and has the advantage of being straightforward and quickly implemented.

Let us suppose that $C(N,m,\varepsilon)$ is the number of points separated by a distance less than ε for a given embedding dimension (Takens,1981), the correlation function[6] is given by

$$C(N,m,\varepsilon) = \frac{1}{N(N-1)} \sum_{m \le t \ne s \le N} H(\varepsilon - \|X_t - X_s\|) \qquad \varepsilon > 0 \tag{1}$$

where $H(z)$ is the Heaviside function given by $H(z) = 1 \ \mathit{for\ all}\ z \ge 0$ and 0 otherwise, ε is the sufficiently small distance between vectors X_t and X_s, and $\|.\|$ is the norm operator.

The correlation function $C(N,m,\varepsilon)$ gives the probability that a randomly selected pair of delay coordinate points is separated by a distance less than ε. It measures the frequency with which temporal patterns are repeated in the data.

To determine the correlation dimension from (1), we have to determine how $C(N,m,\varepsilon)$ changes as ε changes. As ε grows, the value of $C(N,m,\varepsilon)$ grows because the number of near points to be included in (1) increases. Grassberger and Procaccia (1983) show that for sufficiently small ε, $C(N,m,\varepsilon)$ grows at rate D_C and can be well approximated by

$$C(N,m,\varepsilon) \approx \varepsilon^{D_C} \tag{2}$$

That is, the correlation function is proportional to the same power of D_C that represents the value of the correlation dimension.

More formally, the dimension associated with the reconstructed dynamic is given by:

$$D_C = \lim_{\varepsilon \to 0} \frac{\log C(N,m,\varepsilon)}{\log \varepsilon} \tag{3}$$

That is, it is given by the slope of the regression of $\log C(N,m,\varepsilon)$ versus $\log \varepsilon$ for small values of ε and depends on the chosen embedding dimension.

If, as m increases D_C continues to rise, then this relationship is symptomatic of a

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4 See [19, 20, 21, 22].

5 This procedure is based upon the method of delay time coordinates by [23]who showed that this type of reconstruction yields a topological equivalent attractor leaving the dynamic parameters invariant.

6 Example by [24].

stochastic system. If the data are generated by a chaotic system, D_C will reach a finite limit at some relatively small m (saturation point). The importance of the correlation dimension arises from the fact that the minimum number of variables required to model a chaotic attractor is the smallest integer greater than the correlation dimension itself.

The reliability of implementing this algorithm suffers from some problems. Because it is based on the method of delay time coordinates introduced by Takens (1981), the estimates of the embedding dimension and delay time are so crucial that an unfortunate embedding variables choice yields misleading results concerning the dimension of well-known attractors.

Other than the problems associated with these estimates, the correlation dimension suffers from two other problems related to the choice of sufficiently small ε and the norm operator. With the limited length of the data, it will almost always be possible to select sufficiently small ε so that any two points will not lie within ε of each other [25]

Regarding the norm operator, while Brock's [26] theorem gives the conditions under which the correlation function remains independent of the choice of norm even if Kugiumtzis [27] shows the invalid application of this theorem for short noisy time series, such as economic and financial series. Therefore, under such circumstances, the most reliable results are obtained by using the Euclidian norm [28].

Reliability could also be compromised by using short data sets [29, 30]. In fact, in the case of high-dimensional chaos, it will be very difficult to make estimates without an enormous amount of data. This suggests that the correlation dimension can only distinguish low-dimensional chaos from high-dimensional stochastic processes, particularly with economic data. Furthermore, if the fractal dimension is found, the correlation dimension, as in all nonparametric methods, does not provide information about the dynamics of the process that generated it because it does not preserve time-ordering data [31].

2.2 BDS Test

The BDS test[7] introduced by [32] is a non-parametric method based on the correlation function developed by [18], defined in (1), and used to test for serial dependence and nonlinear structures[8] in a time series[9].

The BDS test is not considered to be a direct test for chaos; rather, it is used as a model selection tool to obtain some information about what kind of dependency exists after removing nonlinear dependency from the data.

The standardised residuals extracted from an ARCH-type model are tested for nonlinear dependence. If there is no dependence, the data are not chaotic because the ARCH-type model has captured all nonlinearities [36]; otherwise, the BDS test is applied to residuals

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[7]Subsequent to its introduction, the BBS test was generalized by [33, 34] and more recently, [35] introduced an iterative version of the BBS test.

[8]"There are three particularly well known tests currently in use for testing for nonlinearity: BDS test, White's neural network test and the Hinich bispectrum test, [38], p. 8.

[9]The BDS test incorporates the embedding dimensions, but it assumes the delay time equals 1. See [38, 39] for the problems when fixing delay time to one. Moreover we have to consider The BDS-G test suggested by [39] as a new way for selecting an adequate delay time which allows to obtain a good approximation of the correlation dimension.

to check if the best-fit model for a given time series is a linear or nonlinear model.
The BDS tests the null hypothesis that the variable of interest is independently and identically distributed (IID). Because IID implies randomness, if a series is proved to be IID, it is random [38].
Under the null hypothesis of whiteness, the BDS statistic is obtained by[10]

$$W(N,m,\varepsilon)=\sqrt{N}\,\frac{C(N,m,\varepsilon)-C(N,1,\varepsilon)^m}{\hat{\sigma}(N,m,\varepsilon)} \tag{4}$$

The correlation function asymptotically follows standard normal distribution $N(0,1)$:
$\lim_{N\to\infty} W(N,m,\varepsilon)\sim N(0,1),\ \forall m,\varepsilon$;
$\hat{\sigma}(N,m,\varepsilon)$ is the standard sample deviation of $C(N,m,\varepsilon)-C(N,1,\varepsilon)^m$.
Moving from the hypothesis that a time series is IID, the BDS tests the null hypothesis that $C(N,m,\varepsilon)=C(N,1,\varepsilon)^m$, which is equivalent to the null hypothesis of whiteness against an unspecified alternative.
Hsieh (1991) shows that the BDS test can detect the presence of four types of non-IID behaviours resulting from a non-stationarity of the series: a linear stochastic system (such as ARMA processes), a nonlinear stochastic system (such as ARCH/GARCH processes), or a nonlinear deterministic system, or low-order chaos [40]. If series are IID so that linear or even conditional heteroskedasticity can describe the relations between data, chaotic tests will not be required. However, if this is not the case, investigating the main properties of chaoticity should not be disregarded.
Because it is based on the correlation dimension, the BDS test suffers from the same limitations. In particular, its performance depends on the size of data sets (N) and ε[11], even though [41] showed how the statistics of this test are correctly approximated in finite samples if:
- the number of data N is greater than 500.
- ε lies between 0.5 θ and 2 θ, where θ is the standard deviation of the series.
- the embedding dimension *m* is lower than N/200.
Moreover, it has been found that the BDS test has low power against certain forms of nonlinearity, such as self-exciting threshold AR processes and neglected asymmetry in volatility [42].

2.3 Lyapunov Exponents.

The time series analysis tools described above—the BDS test and the correlation dimension— allow for the distinction between nonlinear systems with a certain degree of complexity and those without, relying on specific features of these systems: nonlinearity for the BDS test and fractal dimension for the correlation dimension.
The Lyapunov exponent may provide a more useful characterisation of chaotic systems because unlike the correlation dimension, which estimates the complexity of a nonlinear

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[10]Example by [24].
[11]To deepen this point see [43].

system, it indicates a level of chaos of the systems, investigating another different, and perhaps more specific, characteristic of chaotic systems: their sensitivity to initial conditions.

This exponent measures average exponential divergence or convergence between trajectories that differ only in having an ‘‘infinitesimally small’’ difference in their initial conditions. If the trajectories remain within a bounded set the dynamic system is chaotic.

To estimate the greatest exponent or Lyapunov characteristic exponent[12] (λ) from experimental or observational data, there are two classes of methods, both based on reconstructing the space state by the delay coordinates methods. The direct methods[13] proposed by [44, 45] based on the calculation of the growth rate of the difference between two trajectories with an infinitesimal difference in their initial conditions and Jacobian methods[14] where data are used to estimate the Jacobians of underlying processes, and to calculate λ from these. [46] proposed a regression method similar in some respects to the test in [47], which involves the use of neural networks to estimate the Jacobians and λ; it is known as the NEGM test. Some remarkable advantages of the Jacobian methods over the direct methods are their robustness to the presence of noise and their satisfactory performance in moderate sample sizes [48].

The general idea on which all methods are based is to follow two nearby points and calculate their average logarithmic rate of separation.

Consider x_0 and $x_0^{'}$ as two points in the state space with distance $\left\| x_0 - x_0^{'} \right\| = \delta_{x_0} << 1$.

Here, δ_{xt} is the distance after *T* iterations between two trajectories emerging from these points; thus, $\delta_{xt} \approx \delta_{x_0} e^{\lambda T}$ where T is the iteration number and λ is the maximal Lyapunov exponent, which measures the average rate of divergence or convergence of two nearby trajectories. This process of averaging is the key to calculating accurate values of λ using small, noisy data sets.

In a system with attracting fixed points of periodic orbit, the distance $\delta x(x_0, t)$ diminishes asymptotically with time. If the system is unstable, the trajectories diverge exponentially for a while but eventually settle down. If the system is chaotic, $\delta x(x_0, t)$ behaves erratically.

Hence, it is better to study the mean exponential rate of divergence of trajectories from two initially close points using the following algorithm:

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[12]“[…] maximal Lyapunov exponent […] is the inverse of a time scale and quantifies the exponential rate by which two typical nearby trajectories diverge in time. In many situations the computation of only this exponent is completely justified, […]. However, when a dynamical system is defined as a mathematical object in a given state space, […] there exist as many different Lyapunov exponents as there are space dimensions”, [49], p. 174.

[13]Some limitations of this methods are highlighted in [48].

[14]To obtain the Lyapunov exponent from observational data, [50, 51] proposed a method, known as the Jacobian method, which is based on nonparametric regression.

$$\lambda = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \ln \frac{|\delta x(x_0, t)|}{|\delta x_0|} \tag{5}$$

The exponents can be positive or negative, but at least one exponent must be positive for an attractor to be classified as chaotic.
In particular, if $\lambda < 0$, the system converges to a stable fixed point or stable periodic orbits. A negative value of the Lyapunov exponent is characteristic of dissipative or non-conservative systems. If $\lambda = 0$, the system is conservative and converges to a stable cycle limit. If $\lambda > 0$, the system is unstable and chaotic. Therefore, if the system is chaotic, it will at least have a positive Lypunov exponent[15]. In fact, one definition of chaotic systems is based on a positive Lyapunov exponent [44, 52, 53]. Finally, if $\lambda = \infty$, the system is random.
Positive Lyapunov exponent is generally regarded as necessary but not sufficient for presence of chaos. As for correlation dimension, the estimate of Lyapunov exponent requires a large number of observations. Since few economic series of such a large size are available, Lyapunov exponent estimates of economic data may not be so reliable.

3 Topological Tools: Recurrence Analysis.

The tests used to detect chaotic structure often fail to find evidence of chaos in aggregated economic data, even if those data are generated by a nonlinear deterministic process. This difficulty is a direct consequence of some problems related to the application of metric and dynamical techniques to economic data. First of all, data quantity and data quality are crucial when applying these techniques, and the main obstacle in empirical economic analysis is addressing short and noisy data sets.
Little or no evidence for chaos has been found in macroeconomic time series. Investigators have found substantial evidence for nonlinearity but relatively weak evidence for chaos per se.
That is due to the small samples and high noise levels for most macroeconomic series; they are usually aggregated time series coming from a system whose dynamics and measurement probes may be changing over time.
In contrast to the laboratory experiments where a large amount of data points can easily be obtained, most economic time series consist of monthly, quarterly, or annual data, with the exception of some financial data with daily or weekly time series.
The analysis of financial time series has led to results which are, as a whole, more reliable than those of macroeconomic series. Financial time-series are a good candidate for analyzing chaotic behaviour. The reason is the much larger available sample sizes and the superior quality of financial data.
Controversial results also arise from using inappropriate analytical methodologies that are more similar to standard statistical protocol. To distinguish between chaotic and non-chaotic behaviours, all researchers, before applying chaos tests, filtered the data using either linear or nonlinear models.

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[15] "[…] the magnitude of the exponent reflecting the time scale on which system dynamics become unpredictable", [44] p. 285.

The filtering procedure was supported by [26], who stated that before testing for a possible nonlinear dependency among the observations, we need to remove all linear correlations that may cause the null hypothesis to be rejected. He also argued that with an infinite amount of noise-free data, possible nonlinear structures should be unaffected by the implementation of a linear filtering process. Removing all linear structure is difficult, but a good approximation can be achieved by using an autoregressive moving average (ARMA) fit to stationary data. With the assumption that the residuals are filtered for linear dependence, it is reasonable to assert that any resulting dependence found in the residuals must be nonlinear. Then when nonlinearity is found, ARCH-type models are applied to detect the source. If unexplained nonlinearity remains, chaos tests are applied. This linear filtering procedure is irrelevant if the data are infinite, noise-free, and stationary, conditions that are not testable for economic and financial data.

But more generally, the open question is whether the chaotic properties of a phenomenon are invariant to linear and nonlinear transformations. It has been proved that linear and nonlinear filters can distort potential chaotic structures [54, 55] and may affect the dimensionality of the original data [54, 56, 57], providing a false indication of chaos. [54] showed that the correlation dimension is not invariant to filtering by the MA (*moving average model*) because, in this way, the fractal structure of the dynamics is lost.

The failure to find convincing evidence for chaos in economic time series redirected the interest to additional tests that work with small data sets and that are robust against noise. This goal seems to be reached by topological tools [58] based on topological invariant testing procedure (close return test and recurrence analysis). Compared to the existing metric and dynamical classes of testing procedures - correlation dimension, the BBS test, and Lyapunov exponent - these tools could be better suited to testing for chaos in financial and economic time series and to provide information about the underlying system responsible for chaotic behaviour [59].

Topological tools are characterised by studying the organisation of the strange attractor because they exploit an essential property of a chaotic system, i.e. the tendency of the time series to nearly, although never exactly, repeat itself over time. This property is known as the recurrence property.

Unlike the metric approach, as the topological method preserves time ordering [31], that's the temporal correlation in a time series in addition to the spatial structure of the data, where evidence of chaos is found, the researcher may proceed to characterise the underlying process in a quantitative way. Thus, one is able to reconstruct the stretching and compressing mechanisms responsible for generating the strange attractor.

Examples of these tests are Close Returns Test and Recurrence Analysis. Both tests consist of two parts. For Close Returns Test we have a qualitative component, that is a graphical representation of the presence of chaotic behaviours - the Close Returns Plot (CRP) - and a quantitative one that tests the null hypothesis that the data are IID against both linear and nonlinear alternatives. It exhibits the same performance as the BDS test even if it detects the recursive behaviour of chaotic time series.

Recurrence Analysis is composed by the Recurrence Plot (RP) developed by [51], the graphical tool that evaluates the temporal and phase space distance, and Recurrence Quantification Analysis (RQA), the statistical quantification of RP [60].

Recurrence Analysis and the Close Returns Test are more similar because they are based on the same methodology but differ in the plot construction. RPs are symmetrical over the main diagonal. Moreover, while the CRP analyses the time series directly and fixes a value ε to estimate nearby points, the RP is based on the reconstruction of time series and an estimation of the points that are close. The starting point of the RPs is based on the time delay method through which the original series is transformed into a set of m-histories.

How the *information* can be recovered from time series was first suggested by [23, 61]. They highlighted that a phase space analogous to that of underlying dynamical system could be reconstructed from time derivative formed from the data. The original series is transformed into an m-dimensional system that, depending on the fulfilment of certain conditions, is topologically equivalent to the original system from which the series was supposedly determined. The one-dimensional signal $X(t)$ is expanded into an m-dimensional phase space by substituting each observation with vector:

$$Y_i = \{x_i, x_{i-d}, x_{i-2d}, ...x_{i-(m-1)d}\} \tag{6}$$

As a result, we have a series of vectors:

$$Y = \{y(1), y(2), y(3), ...y(N-(m-1)d\} \tag{7}$$

where N is the number of observations, *m* is the embedding dimension and *d* is the delay time. If the unknown system that generated $\{x_t\}_{t=1}^{n}$ is N-dimensional, and provided that embedding dimension[16] is $m \geq 2n+1$[17], the set of *m-histories* recreates the dynamics of the data-generating system and can be used to analyse its dynamics [23, 61].

3.1 Recurrence Plot

The Recurrence Plot is a two dimensional representation of those *m-histories* whose coordinates are the present and lagged values of the series.

By using an appropriate norm and fixing a threshold ε [18] that determines if vectors *x(i)* and *x(j)* are sufficiently close together – distance between them below or equal to ε - we obtain a recurrence matrix formally expressed as following:

$$R(i, j) = \Theta(\varepsilon \| x(i) - x(j), \| \qquad for \;\; i, j = M \tag{8}$$

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[16]Basic elements to reconstruct the time series from the original one are the delay-time and the embedding dimension. In the literature there are some techniques like the False Nearest Neighbor and the Mutual Information Function in order to choose respectively the embedding dimension and the delay-time.

[17]According to the numerical results provided by [61] it is possible to get reasonable results with much smaller embedding dimensions. This point is particularly interesting in different economic applications since in such cases the dimension of the true phase space is often not known a priori. Over the years this insight has been widely adopted in economic literature on chaos where common practice is to choose m around 10–12. [16].

[18]A crucial parameter of an RP is the threshold ε. Therefore, special attention has to be required for its choice. See [62]

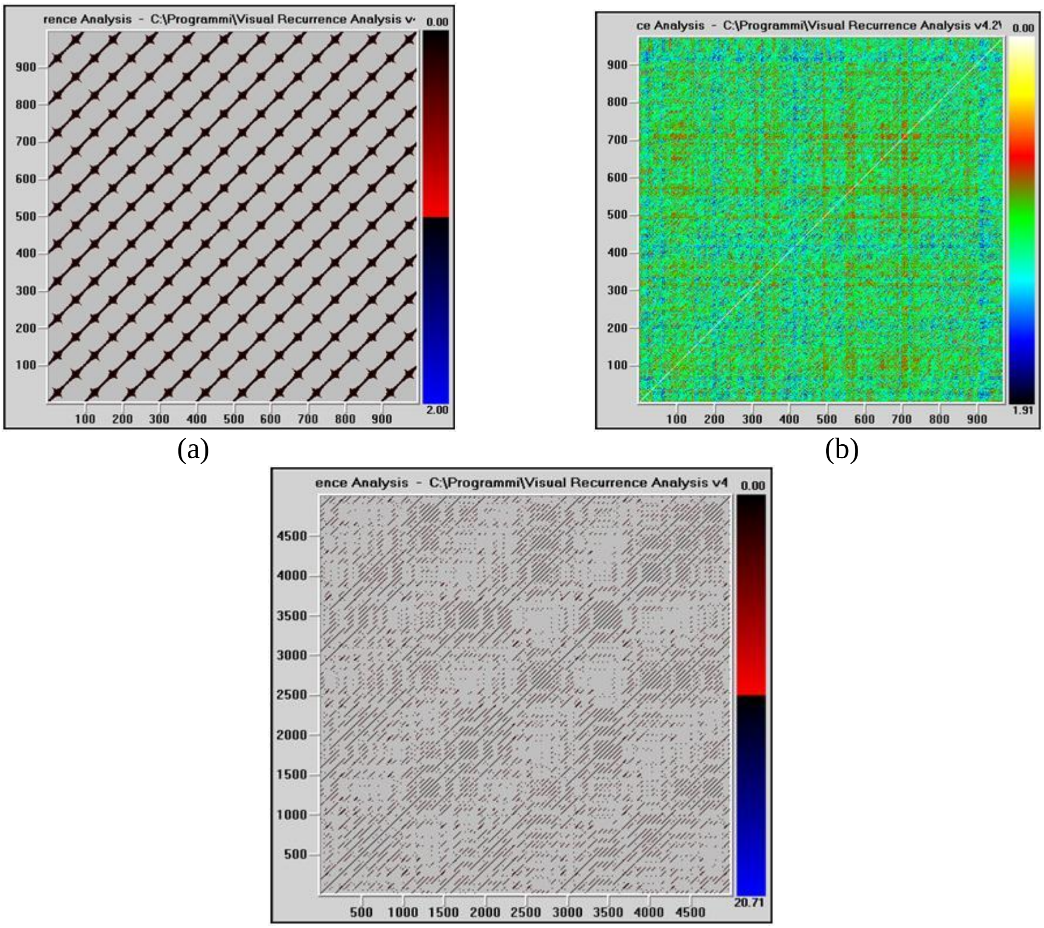
where *M=N-(m-1)d*, Θ is the Heaviside function, and || || is a norm, generally Euclidian[19]. The matrix **R** consists of values 0 (no recurrence) and 1 (recurrence).
More formally:

$$R(i, j) = \begin{cases} 0, & if \quad \|x(i) - x(j) > \varepsilon\| \\ 1, & if \quad \|x(i) - x(j) \le \varepsilon\| \end{cases} \tag{9}$$

The RP is obtained by plotting these binary entries using different colours. Generally dark colour marks nonzero values, that is, short distances, and a light colour zero values, that is, the long distance. Both axes of the RP are time axes and show rightwards and upwards (convention). Since Ri,i ≡ 1 |N i=1 by definition the RP always has a black main diagonal line, the line of identity and it is symmetric with respect to the main diagonal, i.e. Ri,j ≡ Rj,I (Marwan et al., 2007).
The points along the parallels to the 45 degree line are characterized by the same grey tone and this indicates that the couples of observations that keep the same temporal distance are also characterized by the same spatial distance (represented by the same grey tone) [2]
This graphic tool shows different structures depending on the nature of the series under study.
In particular, it is capable of detecting the time recurrence patterns underlying deterministic systems (whether they are chaotic or not). Non-chaotic deterministic systems exhibit very simple regular structures, while the RPs of chaotic systems also show a certain regularity but with more complex and denser features. On the other hand, the RPs obtained from purely random systems do not show distinguishable patterns, appearing as a cloud of points with no apparent structure.
To illustrate the basic ideas behind RP some examples by Visual Recurrence Analysis (VRA)[20] are used. We start by considering a random time series (White noise). The plot (fig. 1b) has been built using delay 1 and dimension 12 as selected respectively from Mutual Information Function (MIF) [63], and False Nearest Neighbours (FNN) [64], both calculated by the software. As we can see in fig.1b, the plot of random time series shows recurrent points distributed in homogenous random patterns - a cloud of points. This means that the random variable lacks deterministic structures. Always in Fig.1 it is possible to characterize stationary and non-stationary processes. If the texture of the pattern within such a block is homogeneous, stationarity can be assumed for the given signal within the corresponding period of time; non-stationary systems cause changes in the distribution of recurrence points in the plot which is visible by brightened areas.
Diagonal structures show (fig. 1c) the range in which a piece of the trajectory is rather close to another piece of the trajectory at different times. From the occurrence of lines

[19]The most frequently used norms are the L1-norm, the L2-norm (Euclidean norm) and the L∞-norm (Maximum or Supremum norm). Note that the neighbourhoods of these norms have different shapes (Fig. 4). Considering a fixed ε, the L∞-norm finds the most, the L1-norm the least and the L2-norm an intermediate amount of neighbours. To compute RPs, the L∞-norm is often applied, because it is computationally faster and allows to study some features in RPs analytically. See [62]
[20]Eugene Kononov http://home.netcom.com/~eugenek/download.html

parallel to the diagonal in the recurrence plot, it can be seen how fast neighboured trajectories diverge in phase space. These lines would not occur in a random as opposed to deterministic process. Thus, if the analysed time series is chaotic, then the recurrence plot shows short segments parallel to the main diagonal: chaotic behaviour causes very short diagonals, whereas deterministic behaviour causes longer diagonals (fig.1a vs. fig.c). If the series is white noise, then the recurrence plot does not show any kind of structure: there are no segments parallel to the main diagonal.



(a) (b)

(c)

Figure 1: Examples by VRA. (a) Periodic time series; (b) White Noise; c) Henon equation

This procedure has some advantages such as simplicity of implementation, robustness to sample length, high dimensionality, noisy dynamics in the underlying equations of motion and fewer prior requirements of the database used [65]. RP analysis is independent of limiting constraints such as data set size, noise, and stationarity; prewhitening of the data (linear filtering, detrending, or transforming the data to conform to any particular distribution) is not necessary as stationarity is not as essential like for the metric approach [66].

Nevertheless some limitations are present. The first one is the construction of RPs and obtaining the Recurrence Matrix (RM). Because they are carried out on the basis of the time delay method, which requires previously fixing the values of the embedding dimension and the time delay, the results obtained from the RP application are sensitive to the values chosen for these parameters [67, 68, 69]. Time delay determines the time separation or predictability of the components in the reconstructed vectors of the system state. It should be chosen so that the elements in the embedding vectors are no longer correlated, thus subsequent analysis would reveal spatial (or geometrical) structures. Since the system is unknown, we estimate optimal time delay as the one where average mutual information (MIF) reaches its first minimum.

The embedding dimension determines the number of the components in the reconstructed vector of the system state. It should be large enough to unfold the system trajectories from self-overlaps, but not too large as the noise will amplify. For a rough selection of the embedding dimension for our one-dimensional time series, we employ the false nearest neighbour (FNN) method suggested by [64].

Even though [70] state that these graphics are independent of these parameters, that is to say, they remain qualitatively stable when the parameters change, other investigations have shown strong evidence to the contrary [71, 72]. [73] (p. 242) conclude that "both groups are in some sense right", showing that some indicators obtained from the RPs are independent from the embeddings while others are not.

The second one is the difficulty to interpret the graphical output of RP. Sometimes the signature of determinism, the set of lines parallel to the main diagonal, might not be so clear. [74, 75] recognised this disadvantage and tried to overcome it by proposing a statistical quantification of RP, which is well known as Recurrence Quantification Analysis (RQA).

3.2 Recurrence Quantification Analysis

The RQA considers that it is possible to quantify the information supplied by RP and, using certain simple pattern recognition algorithms, to summarize the information in a set of indicators or statistics. In this way more objective information than that which could be derived from a purely visual analysis are obtained.

Considering that RP is symmetric, the set of indicators is obtained using the upper or lower triangular part of RP excluding the main diagonal. The main indicators are recurrence rate, determinism, averaged length of diagonal structures, entropy and trend.

Recurrence rate (REC): recurrence points percentage defined as (Aparicio et al. 2008):

$$\%REC = \frac{NREC}{NP} x100 \tag{10}$$

where NREC is the number of recurrent points and NP is the total element of the recurrence matrix. Roughly speaking REC is what is used to compute the correlation dimension of data (Eq. 6).

Determinism rate (DET) is the ratio of recurrence points forming diagonal structures to all recurrence points. DET[21] measures the percentage of recurrent points forming line segments that are parallel to the main diagonal and is calculated as

$$\%DET = \frac{NPD}{NREC} x100 \tag{11}$$

where NPD is the number of points on lines parallel to the main diagonal caused by the existence of time correlation within the trajectory.
The presence of such diagonal structuring in RM is assumed to be a distinctive feature of deterministic structures, absence, instead, of randomness. DET is related with the determinism of the system: the greater the number of points is on line segments, the greater the general dependence of the series will be.

Maxline (MAXLINE) represents the averaged length of diagonal structures and indicates the longest line segments that are parallel to the main diagonal. Unlike the %DET counts all the points on the parallel lines equally regardless of their size, this indicator considers the length of the different lines. It is claimed to be proportional to the inverse of the largest positive Lyapunov exponent. A periodic signal produces long line segments, while the *noise* does not produce any segments. Short segments indicate chaos.

Entropy (ENT) (Shannon entropy) measures the distribution of those line segments that are parallel to the main diagonal and reflects the complexity of the deterministic structure in the system. This ratio indicates the time series *structuredness* so high values of ENT are typical of periodic behaviours, while low values are typical of chaotic behaviours. A high ENT value indicates a large diversity in diagonal line lengths; low values indicate small diversity in diagonal line lengths [77]. "[...] short line max values therefore are indicative of chaotic behaviours" [70, 71]).
The value ***trend*** (TREND) measures the paling of the patterns of RPs away from the main diagonal used for detecting drift and non-stationarity in a time series. It is calculated as a slope of the %REC as a function of the displacement of the main diagonal [78].
In the fig. 1b the visual features are confirmed by the ratios calculated with RQA. We can see that the REC and DET assume values equal to zero, so, in time series there are no recurrent points and no deterministic structures. These features are more evident if we compare REC and DET of time series with one of sine function (fig. 1a). The plots of sine function are more regular and REC shows not only the recurrent point in each epoch but also that this value is the same. DET values are high meaning strong structures in the time series confirmed by the MAXLINE values which are also high, so deterministic rules are present in the dynamics. Comparing fig. 1a and fig. 1c it is possible to see that if the analyzed series is generated from a determinist process in the RP there are long segments parallels to the main diagonal. If the data are chaotic these segments are short.

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[21]"This is a crucial point: a recurrence can, in principle, be observed by chance whenever the system explores two nearby points of its state space. On the contrary, the observation of recurrent points consecutive in time (and then forming lines parallel to the main diagonal) is an important signature of deterministic structuring" [76].

4 International chaos

In their paper [14] analyze the quarterly macroeconomic data of GDP from 1960 to 1988 for West Germany, Italy, Japan and England. The goal was to check for the presence of deterministic chaos. To ensure that the data analysed was stationary they used a first difference[22] then tried a linear fit.

Using a reasonable AR specification for each time series their conclusion was that time series showed different structures. In particular non linear structure was present in the time series of Japan. Nevertheless the application of typical tools for detecting chaos (correlation dimension and Lyapunov exponent) didn't show presence of chaos in any time series. Therefore, although the presence of nonlinearity and correlation dimension values led to admit chaotic behaviour in the time series, Lyapunov exponent test did not support this hypothesis. Probably, as admitted by the same authors, this conclusion could be given by the shortness of series. "With longer time series matters could change"[23].

The conclusion of authors was that none of the countries' income appeared to be well interpreted as being chaotic. The authors ascribe their result to shortness of time series highlighting that with longer time series it could be possible to reach a contrary result. This conclusion is our starting point. By applying Visual Recurrence Analysis[24] we will analyse these time series with the purpose of verifying if the analysis performed by a topological tool[25] could give different results from ones obtained using a metric tool.

The time series[26] chosen were GDP of Japan and GDP of the United Kingdom. The choice was based on the fact that Japan is considered among four of the most dissimilar ones. In fact, in order to filter this series, the authors used an Ar-4, while for the others an Ar-2 was used. For this series they refused the hypothesis IID and the correlation dimension value calculated for various values of M, (the embedding dimension), grew less than the growth of the value of the embedding dimension. There was a saturation point.

In fact for calculated values of M, 5, 10, 15, the dimension of correlation was respectively, 1.3, 1.6, 2.1, against values of 1.2, 3.8, 6.8 of the series shuffled for which aThe rejection of the IID hypothesis, the value of correlation dimension compared with value of shuffled series, and the presence of nonlinearity led the authors to suspect that time series could be chaotic. However, tested with the Lyapunov exponent, the conclusion was that data didn't manifest chaotic behaviour.

In fact the value of the Lyapunov exponent test was negative[27]. They showed that Japan's economy is the most stable of the analysed countries.

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[22] In economics, Gross Domestic Product (GDP) is used to measure of the size of the macroeconomy, but given that this variable is non-stationary and measured using current prices, the main metric for economic expansion that economists use to assess the rate of growth of the macroeconomy is the change in real (inflation-adjusted) GDP over time [79]

[23] [14], p. 1581.

[24] Recurrence Analysis software used for our analysis is Visual Recurrence Analysis by Eugene Kononov. http://home.netcom.com/~eugenek/download.html An extensive survey of different software used to apply these techniques is provided in [15].

[25] An extensive survey of different software used to apply these techniques is provided in [15].

[26] In the analysis performed by [14] the data are for the Japan Real GNP seasonally adjusted, quarterly from 1960 to 1988 and for United Kingdom from 1960 to 1988. Source Datastream.

[27] See "Table 4" p. 1580, in [14].

The GDP time series of United Kingdom was chosen because, as emphasized by the authors, for it, as for Germany, time series is not rejected by the hypothesis IID. The behaviour of correlation dimension is the same for all three European countries[28].

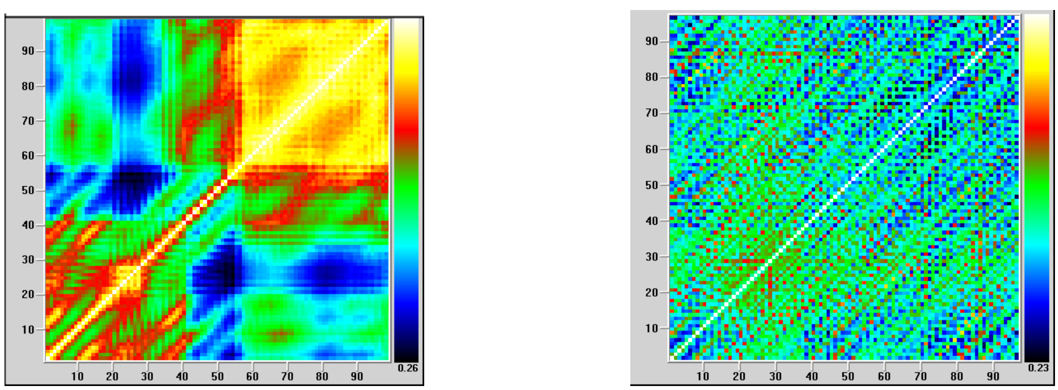
The increase of the embedding *dimension* corresponds to sustained increase of the dimension of correlation. Such increase is also characterised in time series *shuffled* obtained from the time series fits with Ar-2. From this conclusion and considering that the values of Lyapunov exponent test were negative the authors conclude that European time series didn't show non-linearity and in particular chaotic behaviour.

The GDP data of Japan and United Kingdom were analysed with Visual Recurrence Analysis. Following the consideration reported in the paragraph 2.4 and supported by [66] the data were implemented and analysed without prewhitening.

The Recurrence Plot (RP) of Japan GDP is shown in Fig.2a. This was built using a delay-time and embedding dimension respectively equal to 2 and 7. Embedding parameters are determined by the method of false nearest neighbours and the delay by mutual information, and the results are displayed in table 1 for the euro area member states and table 2 for non-euro area countries and the euro area aggregate.

The analysis of VRA using the shuffled series of Japan is described in Fig. 2b.

Comparison between RP of the original time series (Fig.2a) and RP of the shuffled series (Fig. 2b) allows to highlight that the first is non-stationary. The different and diversified colours allow us to support that the more homogenous coloration from the shuffled series (Fig. 2b) is typical of stationary data. Moreover if some nearly continuous lines may be noticed in the plot of shuffled series is due to the random number generator, which is a mathematical algorithm and therefore does not produce purely unstructured time series [2]. The absence of deterministic structure is confirmed by ratios of RQA.



(a) (b)

Figure 2: (a) RP of Japan GDP; (b) shuffled time series

In table 1, RQA results are indicated for both time series: shuffled and not. For the original series REC is positive meaning that the data are correlated. DET is also positive indicating that roughly 43% of the recurrent points are consecutive in time, that is, form

[28]Table 2, p. 1579 in [14].

segments parallel to the main diagonal. This indicates that in the data there is some type of structure. As we saw in fig. 1a long segments indicate that the series is periodic, short segments that the series is chaotic (fig. 1c).
The value of MAXL is 28. This value indicates length of the longer segment in terms of recurrent points of the longer segment and makes it possible to say that the data are non-linear and it is not possible to exclude the presence of chaotic behaviour.

Table 1: RQA Statistics of original and shuffled time series

Japan		
GDP	***1960-1988***	***Shuffled***
Delay	2	2
Dimension	7	8
REC	2.314	0.0
DET	48.485	0.0
ENT	1.00	0.0
MAXL	28	0.0
TREND	-87.39	0.0

The statistics of the RQA (Tab.1) indicate that the shuffled series has lost all information, there are no recurrent points (REC), or segments parallel to the main diagonal (DET). Therefore, no type of deterministic structure is present. This consideration is confirmed by the fact that the value of the MAXL is zero. By comparing original time series with its shuffling we can conclude that the data of Japanese GNP are characterised by non-linearity, confirming the result presented by [14], and they are non stationary.
Our conclusion regarding the presence of chaotic behaviour is different[29]: the data can be chaotic. Therefore, if the authors ascribe the result of their analysis to the shortness of the time series highlighting that with longer time series it could be possible to reach a contrary result[30], the VRA analysis, which can be applied and gives reliable results also with short data sets, shows presence of chaotic behaviour in those data.
In fig. 3 we can see the RP of the United Kingdom GDP. This was built with delay-time and embedding dimension respectively equal to 1 and 8. By comparing the RP of the original time series (Fig. 3a) and its shuffling (Fig.3b) we deduce that the time series is non-stationary: the economy of the United Kingdom is characterised by a period of structural change[31].

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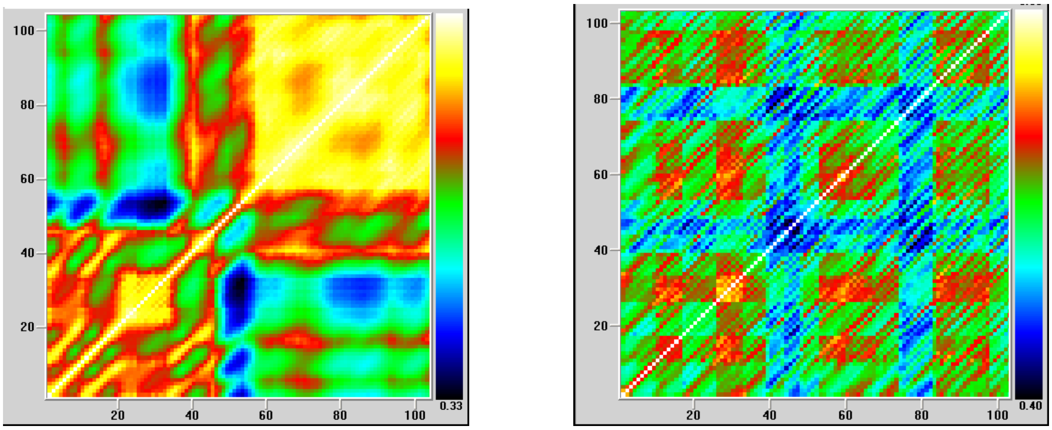
[29]"[…] None of these countries' national income would appear to be well interpreted as being chaotic.", [14] p. 1581.
[30]"[…] When interpreting the findings one must be cautions given the shortness of the series. With longer time series matters could change", [14], p. 1581.
[31]For three decades from 1960, Japan experienced rapid economic growth, which was referred to as the Japanese post-war economic miracle. With average growth rates of 10% in the 1960s, 5% in the 1970s, and 4% in the 1980s, Japan was able to establish and maintain itself as the world's second largest economy from 1968 until 2010, when it was supplanted by the People's Republic of China.



(a) (b)

Figure 3: (a) RP of UK GDP; (b) shuffled time series

Table 2 summarises the statistics of RQA for original time series and its shuffling. The statistics of original time series indicate that in the data there are recurrent points (REC positive), that is, more than 8% of the points that compose the area of the RP's triangle are correlated.

Of this 8%, 32% (DET) shapes segments parallel to the main diagonal, indicating the presence of determinist structures. This conclusion is confirmed by the presence of a positive value of the MAXL. The same ratios of shuffled series are characterised by zero values or negative.

Table 2: RQA Statistics of original and shuffled time series

United Kingdom		
GDP	***1960-1988***	***Shuffled***
Delay	1	1
Dimension	8	9
REC	8.458	0.0
DET	32.009	0.0
ENT	1.972	0.0
MAXL	26	0.0
TREND	77.803	0.0

Comparing our analysis with the one performed by [14] it is possible to highlight some points of difference. While they do not refuse hypothesis IID, the analysis led with VRA induces us to refuse this hypothesis and to emphasize the presence of structure. The data of the United Kingdom are non-linear and this nonlinearity can be interpreted as chaos.

The MAXL and DET value, in fact, confirm that. Also for the United Kingdom as for Japan, [14] emphasized that the analysis carried out on longer series could have obtained different results from the ones we reached. A different conclusion also concerns the fact that, while for Frank et al. the Japanese economy seems more stable than that of the European countries, our analysis (also if limited just to the United Kingdom) is performed from a different point of view. While in Frank M., et al. the comparison was made between stable economies, our analysis is based on unstable economies. The economy of Japan in these years (60-88) is less unstable than that of UK. Testing data sets with Visual

Recurrence Analysis have provided different conclusions from the original work. Our analysis, although performed using a short time series indicates the presence of chaotic behaviour in Japan and United Kingdom time series.

5 Conclusions

The main purpose of this paper has been to illustrate how recurrence analysis can be applied to an important macroeconomic issue, in order to shed light on the chaotic dynamics present in an economic system.

From our analysis compared with the more conventional one by [14] it is possible to conclude that the topological approach can be useful for economic analysis performed on short time series, typical of complex economy, to show the presence of chaotic dynamics. Considering the features of economic time series Recurrence Analysis offers a unique non-linear approach to analysing them.

There are few existing studies of macroeconomic dynamics which utilize this methodology, and so the application in this contribution serves to illustrate the potential of this tool in the study of economic data, but more important to support the conclusion reached by [14] that the data analysed could be chaotic. The conclusion of authors was that none of the countries' income appeared to be well interpreted as being chaotic ascribing their result to shortness of time series and highlighting that with longer time series it could be possible to reach a contrary result. The application of Recurrence Analysis seems to support this contrary result.

Sometimes the conclusions both for and against chaos are reached by applying only one type of chaos test. To produce convincing results, we have to employ all tests for chaos to exploit their different potentials and limits. Few published papers have jointly applied the BDS test, the correlation dimension test, and the test for a positive Lyapunov exponent. Our work is a further example in this direction showing how chaotic behaviour could be detected with a topological tool in the data analysed by [14] only using metric tests.

There are important reasons to understand the impact of nonlinearities and chaos in social systems. First of all it is possible to have a more realistic description of economic phenomena. Most economic variables, whether micro-level, such as prices and quantities, or macro-level, such as consumption, investment and employment, oscillate and it is difficult to find a specific pattern in these oscillations at the level of micro and macro variables because they are not cyclic [80] and not due to external shocks. It is very interesting to use chaos theory for modelling because in this we have a non-explosive system (no trend), an aperiodic system (no seasonality), and a stationary system (invariant distribution ergodicity) [81].

In particular chaotic nonlinear systems can endogenise shocks. Chaotic dynamic models allow for the explanation of persistent and irregular fluctuations without stochastic exogenous shocks introduced ad hoc.

Nevertheless serious limit for a wide application of this approach in Economics is represented by inherent unpredictability of chaotic systems. Concerning this point in Economics we "distinguish between the pessimistic and apathetic approaches, that only pay attention to identify chaos with unpredictability, and those that tries to deep and find the way through which nonlinear dynamic and chaos can help the economy as an evolving complex system. As we say, the first group, that is, the frigid one, principally the econometricians, argues that is not possible the predictability, and because of it chaos theory doesn´t work, but the fact that we cannot make exact predictions of the long-term behaviour of chaotic systems does not exclude the possibility of making, more or less accurate, short-run forecasts" [82].

Even if the future is unknowable, nonetheless Chaos Theory allows for the possibility of a range of future states represented by attractor on which orbits chaotic trajectories evolve. In the long run, a chaotic system moves into, and remains in it, though in principle determinate, resembles a random walk, repeatedly visiting each point in the attractor. The global behaviour of chaotic systems is bounded on the attractor: is not explosive. We can see the attractor as the season trend and the daily data "attracted" to the trend values. This is one way of saying that the daily data are permitted to vary, but are more likely to be close to the trend and the size of these fluctuations from one period to the next have a characteristic probability distribution [83].

While economic fluctuations are unpredictable they will always lie within certain bounds. Thus, if we are able to know in which space the attractor lies, by determining the phase space using the embedding dimension for instance, and if we are able to re-build the orbits, then we can make predictions [81].

Although we cannot forecast the precise state of a chaotic system in the longer term, chaotic systems trace repetitive patterns which often provide useful information [84] because they are the same at different scale of time. What is observed at a more global level is reproduced at a smaller scale because the chaotic attractor is a fractal. So, having knowledge of such patterns would make it possible to, on the average, make better micropredictions[32].

Moreover, exploring economic system by chaotic means is likely to be a new approach helpful for the government to formulate relevant policies to macro-control economy. In the real world where complex dynamics occur ‘‘....linear models are fundamentally wrong or misleading, skewing our understanding of the economy and perhaps corrupting the associated policy advice. It is possible to hold belief in a ‘wrong' theory that generates ‘incorrect' policy advice, in the sense that if the advice is executed, the actual net effects will be fundamentally different from those predicted by the theory" [85, p. 849].

The policies focusing only on one economic variable are condemned to fail. On the contrary, policies that take into account a set of economic variables can be more flexible and consequently efficient, in the sense that they will have higher probabilities of minimizing deviations from their final policy objective" [13].

Nevertheless, chaotic models can be used to suggest ways that people might intervene to achieve certain goals. Chaotic systems could be controlled, that is, made non-chaotic, by manipulating certain variables relating to the particular system.

[32]One computer analysis of stock market data suggests that there are self-similar patterns at 14, 5 and 2 yr. periods and in 5 month periods and that the same patterns may be present within each day. F. D. Peat http://www.fdavidpeat.com/bibliography/essays/chaos.htm

Controlling chaotic systems[33] can be more efficient than controlling linear ones, because only a small push could be needed to engender a big change in the system [86, 87, 88]. In 1990, Ott, Grebogi and York pointed out that chaos could be advantageous in achieving control objectives. Their method involves stabilizing one of the unstable periodic orbits embedded in the chaotic attractor using small time dependent perturbations of a system parameter. Chaotic motion allows this method to work since all of the unstable periodic orbits will eventually be visited. One simply waits until the chaotic motion brings the system near a neighbourhood of the proper unstable periodic orbit, at which time the small control is applied exploiting the sensitivity to initial conditions.

This feature of chaotic systems in economics could have an important insight. Using sensitivity for initial conditions to move from given orbits to other orbits of attractor means choosing different behaviours of systems, that is, a different trade-off of economic policy. Moreover, the employment of an instrument of control in terms of resources to achieve a specific goal of economic policy will be smaller compared to the use of traditional techniques of control. In other words, small, low-cost policy changes could have a large impact on overall social welfare. Realistic modelling, resource saving and choosing among different trade-offs of economic policies (many orbits) could be significant motivations to use chaotic models in economic analysis.

The results of chaos tests do not prove the existence of chaos in all economic variables but are consistent with its existence; in some cases, this could mean only that some economic phenomena are less complex than others and that the economy of a country or simply a single market of an economy is chaotic, not that an economy is as a whole is chaotic.

Economics is inherently dynamic, evolving system. Change is actually constitutive of all sorts of human co-existence/co-operation and social living over the ages. Chaos Theory allows to take in to account these features and to speak about determinism in this context means to say that changes move following a rule and in a well specified range (attractor).

Given these considerations, studies in this area should grow both in size and importance as a field of their own within economics, although the empirical task of extracting evidence of chaotic dynamics from economic time series is objectively more difficult than in the natural sciences. It is very important to realize that despite the probabilistic behavior of a system that naturally limits its predictability, this must not prevented the development of this research area. In this direction the work in progress by [89, 90] that conjecture that to predict at a medium-term horizon or the work by [91] that demonstrated as a reduction of large forecasting errors is produced by making use of system knowledge form the effective Lyapunov exponent. Just to name few.

Moreover, if the best we can do is make short and approximate predictions, then we should be trying to make "parallel" predictions of similar or "surrounding" events.

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[33]"If the governing equations of such systems are exactly known, then one can easily use the OGY or the Pyragas method to design stabilizing controllers for them. However due to uncertainties, determining an exact governing equation for a system is not possible, and the parameters of a system always have some uncertainties. So introducing a robust control strategy for chaos control in such systems seems to be vital. One of the famous nonlinear methods used as a robust control is the sliding mode. Sliding mode control initially developed for continuous time systems. Due to some technical difficulties it cannot be used directly for discrete dynamical systems generated by nonlinear maps" [92]

Instead of looking for ever more accurate models we should be applying the models we have to a range of conditions similar to those we want to predict. The policy makers have to learn to work with models that behave in this way and use them to discover which kinds of conditions seem generally to develop in the same way and which do not. In economic systems it is often not possible to determine the exact time when something will happen. In many cases it is still feasible to say what events are likely to happen and in what sequence. We have to be able to explore possible futures and to detect warning signs of certain kinds of systemic instabilities or systemic shifts (such as critical fluctuations).
As said Henri Poincaré is much better to look farther without having certainty, that don't look anything at all [82].

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