Journal of Computations & Modelling, vol.3, no.4, 2013, 1-10
ISSN: 1792-7625 (print), 1792-8850 (online)
Scienpress Ltd, 2013

On the Continuous Dependence of Solutions of Boundary Value Problems for Delay Differential Equations

Evangelia S. Athanassiadou[1]

Abstract

We prove a general theorem on the continuous dependence of solutions of boundary value of boundary value problems for delay differential equations with a nonlinear problems boundary condition. The proof is based on the continuity of the Brouwer topological degree. Appropriate remarks on the convergence of sequences of functions improve some known results.

Keywords: Delay differential equations, boundary conditions, continuous convergence.

1 Introduction

In this work we consider the most general boundary value problem for delay differential equations. In particular we study boundary value problems of the form

[1] Department of Mathematics, University of Athens, GR-15784, Panepistimiopolis, Athens, Greece. E-mail: eathan@math.uoa.gr

Article Info: *Received* : July 1, 2013. *Revised* : August 30, 2013.
Published online : December 1, 2013.

$$x'(t) = f(t, x_t), \;\; T(x) = a,$$

where f is a vector function, T is a continuous operator and a is constant vector. We prove a general theorem about continuous dependence of solutions of the above boundary value problem.

Existence, uniqueness and continuous dependence of solutions of boundary value problems of this type have been proved in [1], [2] and [3]. Corresponding results for boundary value problems for ordinary differential equations are included in [4], [5] and [6].More details for problems of this type can be find in the books [7], [8], [9] for ordinary differential equations and [7],[10], [11] and [12] for delay differential equations.

The proof here is quite different from the method, in the papers [1] and [2], where the Schauder's theorem is employed. In our work the continuity of the Brouwer topological degree is applied [13], [14]. In [1] the results are proved in the space of continuous functions but in [2] this space have been replaced by the space of Lipschitzian functions.

In [1] and [2] the theorems require the hypothesis of "unrestricted uniqueness" that is a condition which fulfills the function f such that the corresponding boundary value problem has exactly one solution. This condition is needed only for the limit problem. In this paper we will use the continuous dependence of the Brouwer topological degree [13], [14]. Also we will apply some properties of the convergences of sequences of functions in order to generalize known results.

2 Preliminaries and notations

Let τ be a positive number. The space of all continuous functions

$$\varphi: [-\tau, 0] \to \mathbb{R}^N,$$

will be denoted by $C^0 = C^0([-\tau, 0], \mathbb{R}^N)$ endowed with the supremum norm

$$\|\varphi\|_{C^0} = \sup\{|\varphi(t)|: t \in [-\tau, 0]\}.$$

For a function $x:[-\tau,b]\to\mathbb{R}^N, b>0$and $t\in[0,b]$, we define the function $x_t:[-\tau,0]\to\mathbb{R}^N$ by $x_t(s)=x(t+s), s\in[-\tau,0]$. Especially the condition $x_0=\varphi$ is equivalent to $x(s)=\varphi(s)$, $s\in[-\tau,0]$.

Let $A=A([0,b],\mathbb{R}^N)$be the space of absolutely continuous functions from $[0,b]$ into $\mathbb{R}^N$ endowed with the supremum norm. We denote by $C(A,\mathbb{R}^N)$ the space of continuous operators from A to $\mathbb{R}^N$. We say that a function $f:[0,b]\times C^0\to\mathbb{R}^N$ satisfies the Carathéodory conditions if the following are valid:

(i) for every fixed φ, f is measurable with respect to t,

(ii) for every fixed t, f is continuous with respect to φ and

(iii) for every bounded set $D\subset C^0$ there exists an integrable function m such that

$$|f(s,\varphi)|\leq m(s),$$

for$s\in[0,b],\varphi\in D$. A family Φ of the functions $f:[0,b]\times C^0\to\mathbb{R}^N$ we say that satisfies the Carathéodory conditions uniformly if every function f fulfils the conditions (i), (ii) and also

(iv) for every bounded set $D\subset C^0$ there exists an integrable function Msuch that

$$|f(s,\varphi)|\leq M(s)$$

for every $s\in[0,b]$, $\varphi\in D$ and $f\in\Phi$.With $\mathcal{C}$ we will denote the space of functions

$$f:[0,b]\times C^0\to\mathbb{R}^N$$

which satisfy the Carathéodory conditions.

Now we present some new results regarding $a-$convergence or continuous convergence that we will need.

Let $(X,d),(Y,\rho)$ be arbitrary metric spaces. In particular $X=[0,b]\times C^0 Y=\mathbb{R}^N$ in our case. Also for the remaining of this section, let $f_n, f:X\to Y, n=1,2,\ldots$. We recall the following definitions (See also [15], [16]).

(a) We say that (f_n)*converges* a to $f\left(f_n \overset{a}{\rightarrow} f\right)$ iff, for each $x \in X$ and for each sequence (x_n)in X, with $x_n \rightarrow x$ it holds that $f_n(x_n) \rightarrow f(x)$.

(b) We say that the sequence (f_n) is *exhaustive*, iff

$$\forall\, x \in X\ \ \forall\, \varepsilon > 0\ \exists\, \delta = \delta(x,\varepsilon) > 0\ \exists\, n_0 = n_0(x,\varepsilon):$$
$$d(x,t) < \delta \Rightarrow \rho\big(f_n(x), f_n(t)\big) < \varepsilon,\ \ for\ n \geq n_0.$$

(c) We say that (f_n) is *weakly-exhaustive*iff,

$$\forall\, x \in X\ \ \forall\, \varepsilon > 0\ \exists\, \delta = \delta(x,\varepsilon) > 0:$$
$$d(x,t) < \delta \Rightarrow \exists\, n_t \in \mathbb{N}: \rho\big(f_n(x), f_n(t)\big) < \varepsilon,\ \ for\ n \geq n_t.$$

Obviously if (f_n) is exhaustive then (f_n) is weakly-exhaustive. It is not hard to see that the inverse implication fails ([15]).

Now, we formulate some new results on a −convergence. The proofs of Propositions 2.1 and 2.2 can be found in [15] and the proof of Propositions 2.3 and 2.4 in [16].

Proposition 2.1 *The following are equivalent.*

(a) $f_n \overset{a}{\rightarrow} f$,

(b) (f_n)*convergespointwise to*f*and*(f_n) *is exhaustive.*

Proposition 2.2 *Suppose that*(f_n)*converges pointwiseto*f. *Then the following are equivalent.*

(a) f*is continuous,*

(b) (f_n)*is weakly exhaustive.*

We note that the functions f_n, $n = 1,2, \ldots$ need not to be continuous in the above theorem. Also as a corollary from proposition 2.1, 2.2 we get that the a −limit of any sequence of functions is necessarily continuous. With the next theorems we see how a −convergence and uniform convergence are related.

Proposition 2.3 *Suppose that*$f_n \overset{a}{\rightarrow} f$. *Then*$(f_n)$*converges uniformly to* f *on compact subsets of* X.

For details and concrete examples regarding the difference of a −convergence and uniform local convergence see [16].In the inverse direction, the continuity of the functionsf_n, $n = 1,2, \ldots$is necessary.

Proposition 2.4 *Suppose that* $\{f_n\} \subseteq C(X,Y) \coloneqq \{f: X \to Y | f \text{ is continuous}\}$. *If for each* $x \in X$ *there is a neighborhood* A *of* x *such that* (f_n) *converges uniformly to* f *on* A, *then,*

$$f_n \xrightarrow{a} f.$$

In case that X is locally compact and $\{f_n\} \subseteq C(X,Y)$ as a corollary from propositions 2.3 and 2.4 we get that

$$f_n \xrightarrow{a} f \Leftrightarrow (f_n) \text{ converges uniformly on compacta to } f.$$

In view of the above propositions, some comments are in order:

(a) Suppose $\{f_n : n = 1,2, \ldots\} \subseteq \mathcal{C}$ and that $f_n \xrightarrow{a} f$. Then by propositions 2.1 and 2.2 it follows that f is continuous, hence $f \in \mathcal{C}$.

(b) In theorems on continuous dependence of solutions e.g. theorem 5.1 of Hale [11], we require that $f_n(t,\cdot) \xrightarrow{a} f(t,\cdot)$ for each $t \in [0,b]$. From propositions 2.1 and 2.2, it follows again that $f(t,\cdot)$ is continuous for each $t \in [0,b]$. Also, since $f_n(t,\varphi) \to f(t,\varphi)$, $n \to \infty$ we get that $f(\cdot,\varphi)$is measurable for each $\varphi \in C^0$. Hence condition (i) and (ii) are automatically satisfied by f.

3 Continuous dependence

In this section we prove a general theorem on continuous dependence of solutions of boundary value problems for differential equations with delay.

We consider the following boundary value problem

$$x'(t) = f(t, x_t), \tag{1}$$

$$T(x) = a, \tag{2}$$

where $f \in \mathcal{C}$ and $T \in \mathcal{L}$.

The continuous function $x: [-\tau, b] \to \mathbb{R}^N$ and $a \in \mathbb{R}^N$ is called the solution of (1) in Carathéodory sense if x satisfies (1) almost everywhere on $[0, b]$, x is constant on $[-\tau, 0]$ and x is absolutely continuous on $[0, b]$. The same function is called the solution of boundary value problem $(1), (2)$ if this is a solution of equation (1) in Carathéodory sense and satisfies boundary condition (2).

From Theorem 3.1, Theorem 5.1 and §7 of Hale [11] (see also Theorem 3.2 of [8]) with some modifications we get the following Theorem 3.1 and Theorem 3.2.

Theorem 3.1 *If $f \in \mathcal{C}$, then for each $(t, \varphi) \in [0, b] \times C^0$ with φ constant, there is a solution of (1) passing through (t, φ).*

Theorem 3.2 *Let $f, f_n \in \mathcal{C}$, $\varphi, \varphi_n \in A$, $n = 1,2, \ldots$ with $\varphi_n \xrightarrow{p.w} \varphi$, $\{f_n : n \in \mathbb{N}\}$ satisfies uniformly Carathéodory conditions and $f_n \xrightarrow{a} f$. If x^n is any solution of the problem $x'(t) = f_n(t, x_t)$, $x_0 = \varphi_n$, and the problem $x'(t) = f(t, x_t)$, $x_0 = \varphi$ has unique solution, let x, then x^n converges uniformly to x.*

Now we prove a general theorem on the continuous dependence of boundary value problems for delay differential equations of type $(1), (2)$.

Theorem 3.3 *Let $f_0, f_n \in \mathcal{C}$ and $T_0, T_n \in C(A, \mathbb{R}^N)$, $n = 1,2, \ldots$. Also we suppose that $\{f_n : n \in \mathbb{N}\}$ satisfies uniformly Carathéodory conditions.*

(i) $f_n \xrightarrow{a} f_0$ *and* $T_n \xrightarrow{a} T_0$.

(ii) *The boundary value problem*

$$x'(t) = f_n(t, x_t), \quad x_0 = u \ (constant\ function),$$

where $u \in \mathbb{R}^N$, has at most one solution for every $u \in \mathbb{R}^N$, $n = 0,1,2 \ldots$

(iii) *The boundary value problem*

$$x'(t) = f_0(t, x_t), \;\; T_0(x) = r,$$

has at most one solution for eachr $\in \mathbb{R}^N$.

Let $v, v_n \in \mathbb{R}^N$ *with* $lim_{n\to\infty} v_n = v$. *If* $x^0 \equiv x^0(t, f, v, T)$ *is the solution of*

$$x'(t) = f(t, x_t), \;\; T(x) = v,$$

then for each $\varepsilon > 0$ *there exists* $n_0 = n_0(\varepsilon)$ *such that for* $n \geq n_0$ *the boundary value problem*

$$x'(t) = f_n(t, x_t), \;\; T_n(x) = v_n$$

has a solution $x^n \equiv x^n(t, f_n, v_n, T_n)$ *satisfying*

$$\|x^n - x^0\| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. We consider the following problem

$$\big(P_n(u)\big): \;\; x'(t) = f_n(t, x_t), \;\; x_0 = u,$$

where $u \in \mathbb{R}^N$, $n = 0,1,2, \ldots$.

Assertion:

There exist a neighborhood V *of* $x^0(0)$ *in* $\mathbb{R}^N$ *and* $m \in \mathbb{N}$ *such that for every* $u \in V$, $n \geq m$ *the problem* $\big(P_n(u)\big)$ *has a solution* u^n *satisfying*

$$\|u^n - x^0\| < \varepsilon.$$

Proof of Assertion:

Suppose not, then in view of Theorem 3.1, it follows that for each m, there is $u_m \in B\left(x^0(0), \frac{1}{m}\right)$ and $n_m \geq m$ such that the solution u^{n_m} of problem $\left(P_{n_m}(u_m)\right)$ satisfies $\|u^{n_m} - x_0\| \geq \varepsilon$. But, since $u_m \to x^0(0)$, as $m \to \infty$, it follows from Theorem 3.2 that u^{n_m} converges uniformly to x^0. Hence, we arrive to a contradiction.

We observe that if $f_n = f_0$ for $n = 1,2, \ldots$ then the conclusion of the assertion holds for the solutions of $P_0(u)$ on some neighbourhood of $x^0(0)$.

Now let (V, m) be a pair satisfying the Assertion where V satisfies also the above observation. For each $u \in V$ we set σ_n^u to be the solution of problem $\big(P_n(u)\big)$ such that

$$\|\sigma_n^u - x^0\| \leq \varepsilon \ for \ n \geq m, \ or \ n = 0. \tag{3}$$

We fix a ball B_1 with centre$x^0(0)$ in $\mathbb{R}^N$ such that $\overline{B_1} \subseteq V$ and define

$$F_n: \overline{B_1} \to \mathbb{R}^N, F_n(u) = T_n(\sigma_n^u)(n = 0,1, \dots).$$

We observe that, if $u_n \to u_0$, $u_n \in \overline{B_1}, n = 1,2, \dots$ then by Theorem 3.2 it follows that $\sigma_n^{u_n}$ converges uniformly to $\sigma_0^{u_0}$.

Hence by hypothesis (i) we get that

$$F_n \xrightarrow{a} F_0 ,$$

and Proposition 2.3 implies the uniform convergence of F_n to F_0,

$$F_n \xrightarrow{u} F_0. \tag{4}$$

Also by the uniqueness of solution (hypothesis (ii)), it follows again from Theorem 3.2 that each $F_n(n = 0,1, \dots)$ is continuous (since the mapping $u \to \sigma_n^u$ is continuous). □

In view of (iii), F is injective and we have $v \notin F_0(\partial B_1)$. Hence

$$dist(F_0(\partial B_1), v) = d > 0,$$

since$F_0(\partial B_1)$ is closed. Taking into account that $v_n \to v$ and (4), we get the existence of $n_0 \in \mathbb{N}$, such that

$$dist(F_n(\partial B_1), v_n) \geq d/3,$$

for $n \geq n_0$. Consequently, the Brouwer topological degree $\deg(F_n, B_1, v_n)$ is well defined for $n \geq n_0$ and in view of the continuous dependence of topological degree there is $n_1 \in \mathbb{N}$, $n_1 \geq n_0$ such that $\deg(F_n, B_1, v_n) = \deg(F_0, B_1, v)$ for $n \geq n_1$. Since F_0 is injective, we have (see [13], [14]) $\deg(F_0, B_1, v) = \pm 1$. Hence for $n \geq n_1$ we have

$$\deg(F_n, B_1, v_n) = \pm 1$$

for$n \geq n_1$and the equation $F_n(u) = v_n$ has at least one solutionu_n in B_1. From property (3) of the solution of the problem $\left(P_n(u_n)\right)$ it follows that the solution $x^n := \sigma_n^{u_n}$fulfils the condition $\|x^n - x\| < \varepsilon$, that means the Theorem is proven by taking $n_0 = n_1$. □

Now from Theorem 3.2 we have the following.

Corollary 3.3 *Under the assumptions of Theorem 3.2 we assume additional that the boundary value problem*

$$x'(t) = f_n(t, x_t),\ \ T_n(x) = u_n$$

has a unique solution x^n*. Then we have* $lim_{n\to\infty} x^n = x$ *uniformly on* $[0, b]$.

Proof. It follows directly from Theorem 3.3. □

Corollary 3.4 *If the problems corresponding to equation* $x'(t) = f(t, x_t)$ *have local uniqueness and if the boundary value problem*

$$x'(t) = f(t, x_t),\ \ T(x) = v \tag{5}$$

have at most one solution for every $v \in \mathbb{R}^N$*, then the set* V *of all* $v \in \mathbb{R}^N$ *for which* (5) *has at most one solution is an open subset of* $\mathbb{R}^N$.

Proof. If $v_0 \in V$ it is not an interior point of V, then there exists a sequence (v_n), $n = 1,2, \dots$ in $\mathbb{R}^N$ such that $\lim_{n\to\infty} v_n = v$ and

$$x'(t) = f(t, x_t),\ \ T(x) = v_n$$

has no solution. This contradicts the conclusion of Theorem 3.3 by taking $f_n = f_0$, $T_n = T_0$. □

References

[1] K. Twardowska, On existence, uniqueness and continuous dependence of solutions of boundary value problems for differential equations with a delayed parameter, Zeszyty Naukowe UJ, Prace Mat.,**18**, (1977), 143-153.

[2] M. Jablonski and K. Twardowska, *On boundary Value problems for differential equations with a retarded argument*, 1987.

[3] J.W. Myjak, A boundary value problem for non-linear differential equations with a retarded argument, *Ann. Polon. Math.*, **27**, (1973), 133-142.

[4] S.N. Chow and A. Lasota, On boundary value problems for ordinary differential equations, *J. Diff. Eqs*., **14**, (1973), 326-337.

[5] G. Vidossich, On the continuous dependence of solutions of boundary value problems for ordinary differential equations, *J. Diff. Eqs*., **82**, (1989), 1-14.

[6] G.A. Klaasen, Continuous dependence for N-point boundary value problems, *SIAM J. Appl. Math*, **29**, (1975), 99-102.

[7] J. Mawhin,: *Nonlinear boundary value problems for ordinary differential equations: From Schauder theorem to stable homotopy*, in "Nonlinear Analysis, Academic Press, New York, 1978.

[8] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.

[9] L.C. Piccinini, G. Stampacchia and G. Vidossich, *Ordinary Differential Equations in* $\mathbb{R}^N$ (problems and methods), Springer-Verlag, New York, 1984.

[10] R.D. Driver, *Ordinary and Delay Differential Equations*, Springer-Verlag New York, 1977.

[11] J. K. Hale, Functional differential equations, *Applied Mathematical Sciences*, **3**, Springer-Verlag, New York, 1971.

[12] O. Diekmann, S. A. Van Gils, S. M. VerduynLunel and H. O. Walther, *Delay Equations: Functional-Complex and Nonlinear Analysis*, New York, 1995.

[13] N. G. Lloyd, *Degree Theory*, Cambridge Univ. Press, Cambridge, 1978.

[14] Qian Min and Li Zhengyuan, *Rotating Vector-Field and Degree Theory*, Publishing House of Beijing University, 1982.

[15] V. Gregoriades and N. Papanastassiou, A notion of exhaustiveness and Ascoli-type theorems, *Topology and its Applications*, **155**, (2008), 1111-1128.

[16] E. Athanassiadou, C. Papachristodoulos and N. Papanastassiou, a and Hyper$-a$ convergence in Function Spaces, (submitted).