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Bivariate Copulas-Based Models for Communication Networks

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Abstract

Copulas are functions that join or "couple" multivariate distributions to their one-dimensional marginal distribution functions. In words, copulas are multivariate distributions whose margins are uniform on the interval (0,1). In the present article, we restrict our attention to bivariate copulas and more precisely we discuss the Ali-Mikhail-Haq bivariate model. The special case of the aforementioned model with logistic margins is studied in detail and closed formulas for its basic characteristics are derived. In addition, reliability properties for systems with two exchangeable logistic components are established.

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1. Introduction

The study of copulas and their applications in the field of Probability and Statistics has attracted a lot of research interest in the last two decades. Hutchinson and Lai (1990) were among the early authors who popularized the notion of copulas. Furthermore, Nelsen (2006) studied in detail the bivariate copulas, while Cherubini, Luciano and Vecchiato (2004) illustrated interesting applications of copulas in the field of Insurance and Finance.

Copulas are functions that join or “couple” multivariate distributions to their one-dimensional marginal distribution functions. Alternatively, copulas are multivariate distributions whose one-dimensional margins are uniform on the interval (0,1). Copulas offer also a way to produce scale-free measures of dependence or construct families of bivariate distributions. For more details about the development and study of copulas-based distribution models, the interested reader is referred to the excellent monographs of Nelsen (2006) or Balakrishnan and Lai (2009).

In the present article, we study the bivariate Ali-Mikhail-Haq distribution model. The copula function of the model is presented, while some results for the case of logistic margins are discussed. In Section 3, exact formulae are derived for the basic characteristics of the model, such as the joint cumulative density function or the bivariate survival odds ratio, when $\theta = 1$. In Section 4, the general results presented previously in this paper are exploited in order to reach conclusions referring to lifetime of communication networks.

2. The Ali-Mikhail-Haq copula

Generally speaking, let X, Y be two random variables with corresponding cumulative distribution functions

$$F(x) = P(X \le x) \text{ and } G(y) = P(Y \le y),$$

while $H(x,y) = P(X \le x, Y \le y)$ denotes their joint distribution function. Each pair (x,y) of real numbers leads to a point $(F(x), G(y))$ in the unit square and this ordered in turn corresponds to a number $H(x,y)$ in [0,1]. This correspondence, which assigns the value of joint distribution function to each ordered pair of values of the individual distribution functions, is indeed the function called *copula*. In terms of random variables, let H be a joint distribution with margins F, G. Then, there exists a copula C such that for all values (x,y)

$$H(x,y) = C(F(x), G(y)). \qquad (1)$$

It is worth mentioning that in case of continuous margins F, G, the copula C is unique.

Let us now restrict our attention to the Ali-Mikhail-Haq distribution model. For $u, v \in (0,1)$ and a design parameter $\theta \in [-1,1]$, the Ali-Mikhail-Haq copula is defined as follows

$$C_\theta(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}. \qquad (2)$$

Note that the Ali-Mikhail-Haq model can be equivalently expressed as

$$C_\theta(u,v) = uv\sum_{k=0}^{\infty}\left(\theta(1-u)(1-v)\right)^k.$$

This class of distributions was first introduced by Ali, Mikhail and Haq (1978) and since then it has attracted some research attention (see, e.g. Genest and MacKay (1986) or Mikhail *et al.* (1987)). The following proposition offers some general results for the aforementioned copula.

Proposition 1. (i) The Ali-Mikhail-Haq copula function (defined in (2)) is positively ordered.

(ii) If C_a, C_β are two members of the Ali-Mikhail-Haq distribution family (defined in (2)), then the harmonic mean of C_a, C_β belongs also to the Ali-Mikhail-Haq copulas.

Proof. (i) A parametric family { C_θ } of copulas is said to be positively ordered if $C_a \prec C_\beta$ whenever $a \le \beta$. For the distribution model defined in (2), the following ensues

$$\frac{C_a}{C_\beta} = \frac{1-\beta(1-u)(1-v)}{1-a(1-u)(1-v)} \le 1,$$

where $-1 \le a \le \beta \le 1$ and $u, v \in (0,1)$. Therefore, the desired result is straightforward.

(ii) Let C_a, C_β denote two copulas that belong to the Ali-Mikhail-Haq family defined in (2). Therefore, we have

$$C_a(u,v) = \frac{uv}{1-a(1-u)(1-v)},\ C_\beta(u,v) = \frac{uv}{1-\beta(1-u)(1-v)}.$$

The harmonic mean of C_a, C_β can be expressed as

$$\frac{2}{1/C_a + 1/C_\beta} = \frac{2C_a C_\beta}{C_a + C_\beta} = \frac{2uv}{2-(a+b)(1-u)(1-v)} = C_{(a+\beta)/2}.$$

It goes without saying that the harmonic mean is a Ali-Mikhail-Haq copula function with parameter $\theta = (a+\beta)/2$. □

3. The Ali-Mikhail-Haq model with logistic margins

Let us denote by T_1, T_2 two random variables with corresponding copula function that belongs to the Ali-Mikhail-Haq family (defined in (2)). Let us next assume that the marginal distributions F, G are logistic, namely

$$F(t_1) = \left(1+e^{-t_1}\right)^{-1},\ G(t_2) = \left(1+e^{-t_2}\right)^{-1}.$$

The joint cumulative distribution function of T_1, T_2 is given as

$$H_\theta(t_1,t_2) = \left(1+e^{-t_1} + e^{-t_2} + (1-\theta)e^{-t_1-t_2}\right)^{-1}. \tag{3}$$

It is of some research interest to shed light on the special case $\theta = 1$. In this case, the general model reduces to the well-known Gumbel bivariate logistic distribution (see Gumbel (1961)). Indeed, the joint distribution function takes now the following form

$$H(t_1, t_2) = \frac{1}{1 + e^{-t_1} + e^{-t_2}} \ . \tag{4}$$

Note that may one apply equation (2) and replace u, v with the logistic margins F, G, the following result comes true

$$C_1(F(t_1), G(t_2)) = H(t_1, t_2) \, .$$

The last equation verifies the well-known Sklar's Theorem for the bivariate distribution mentioned above. (Sklar (1959)).

The notion of *odds for survival* for a random variable *X*, namely the ratio $P(X \geq x) / P(X \leq x)$ is of some importance, especially when the random variable expresses the lifetime of a component. Analogously, the *bivariate survival odds ratio* of two random variables X, Y is defined as $P(X \geq x \ or \ Y \geq y) / P(X \leq x, Y \leq y)$. The following remark offers some expressions for the bivariate survival odds ratio of T_1, T_2 defined earlier.

Remark 1. If T_1, T_2 denote two random variables with bivariate Gumbel logistic distribution (defined in (4)), then the *bivariate survival odds ratio* of T_1, T_2 is given as

(i) $$\frac{P(T_1 \geq t_1 \ or \ T_2 \geq t_2)}{P(T_1 \leq t_1, T_2 \leq t_2)} = e^{-t_1} + e^{-t_2}$$

(ii) $$\frac{P(T_1 \geq t_1 \ or \ T_2 \geq t_2)}{P(T_1 \leq t_1, T_2 \leq t_2)} = \frac{P(T_1 \geq t_1)}{P(T_1 \leq t_1)} + \frac{P(T_2 \geq t_2)}{P(T_2 \leq t_2)} \, .$$

Proof. (i) Since the following holds true

$$\frac{P(T_1 \geq t_1 \ or \ T_2 \geq t_2)}{P(T_1 \leq t_1, T_2 \leq t_2)} = \frac{1 - H(t_1, t_2)}{H(t_1, t_2)}$$

the desired result is deduced by employing equation (4).

(ii) The result we are chasing for, is straightforward by recalling that the univariate survival odds ratio of each of the random variables T_1, T_2 takes on the following form

$$\frac{P(T_i \geq t_i)}{P(T_i \leq t_i)} = \frac{R_i(t_i)}{1 - R_i(t_i)} = e^{-t_i},\ i = 1,2,$$

where $R_i(t_i) = e^{-t_i}/(1 + e^{-t_i})$ denotes the survival function of $T_i,\ i = 1,2.$ □

The following proposition offers an expression for the conditional survival function of T_1 given $T_2 = t_2$.

Proposition 2. If T_1, T_2 denote two random variables with bivariate Gumbel logistic distribution (defined in (4)), then the conditional survival function of T_1 given $T_2 = t_2$ satisfies the following

$$P(T_1 > t_1 | T_2 = t_2) = \frac{e^{2t_2} + 2e^{t_1+t_2}(1 + e^{t_2})}{\left(e^{t_1} + e^{t_2} + e^{t_1+t_2}\right)^2} \ . \tag{5}$$

Proof. The conditional survival function of T_1 given $T_2 = t_2$ is defined as

$$P(T_1 > t_1 | T_2 = t_2) = \int_{t_1}^{\infty} f(u|t_2) du \ . \tag{6}$$

Note that $f(u|t_2)$ denotes the conditional probability density function of T_1 given $T_2 = t_2$ and can be expressed via

$$f(u|t_2) = \frac{f(u, t_2)}{f_{T_2}(t_2)},$$

where $f(u, t_2)$ is the joint probability density function of T_1, T_2 while $f_{T_2}(t_2)$ is the probability density function of T_2. Since

$$f(u, t_2) = \frac{2e^{-u-t_2}}{\left(1 + e^{-u} + e^{-t_2}\right)^3}$$

and

$$f_{T_2}(t_2) = \frac{e^{-t_2}}{\left(1+e^{-t_2}\right)^2}$$

the integral appeared in (6), takes on the following form

$$P(T_1 > t_1 | T_2 = t_2) = \int_{t_1}^{\infty} \frac{2e^{-u}(1+e^{-t_2})^2}{(1+e^{-u}+e^{-t_2})^3} du$$

and the proof is complete. □

4. Applications to Communication networks

In this section, we shall present some results based on the copulas-based model described earlier, but now in the framework of communication networks. More specifically, let us consider a structure (network) that consists of two exchangeable components (units) with lifetimes T_1, T_2 respectively. The components are assumed to have bivariate Gumbel logistic distribution, namely (T_1, T_2) are associated to the Ali-Mikhail-Haq model with $\theta = 1$. In the sequel, we study reliability properties of a series and a parallel communication system that consists of bivariate Gumbel logistic components (T_1, T_2). A similar study has been already accomplished in the literature for different bivariate models (see, e.g. Navarro, Ruiz and Sandoval (2008) or Eryilmaz (2012)).

The next proposition offers explicit formulas for the survival (reliability) functions of series and parallel structures with two logistic components.

Proposition 3. Let $T_{(1)}, T_{(2)}$ denote the lifetimes of a series and a parallel system with two components T_1, T_2 respectively. If (T_1, T_2) follow the bivariate Gumbel logistic distribution (defined in (4)), then

(i) the reliability function of the parallel system $T_{(2)}$ is given as

$$R_{(2)}(t) = \frac{2}{2+e^{t}}, \tag{7}$$

(ii) the reliability function of the series system $T_{(1)}$ parallel system is given as

$$R_{(1)}(t) = \frac{2}{2+3e^{t}+e^{2t}} \quad . \tag{8}$$

Proof. (i) Since

$$R_{(2)}(t) = P(T_{(2)} > t) = P(T_1 > t \text{ or } T_2 > t) = 1 - H(t,t)$$

the desired result is effortlessly reached by recalling equation (4).

(ii) Recalling the following well-known equality

$$R_{(1)}(t) = 2R_i(t) - R_{(2)}(t), \quad i = 1{,}2$$

(see, e.g. Baggs and Nagaraja (1996)), the proof is complete.

□

It is of some research interest to study the failure rate of the abovementioned reliability structures. Generally speaking, if X is an absolutely continuous random variable with reliability function $R(x)$ and probability density function $f(x)$, the univariate failure rate is defined as

$$r(x) = \frac{f(x)}{R(x)} \tag{9}$$

for all x such that $R(x) > 0$ (see, e.g. Kuo and Zuo (2003) or Triantafyllou and Koutras (2008)).

Proposition 4. Let $T_{(1)}, T_{(2)}$ denote the lifetimes of a series and a parallel system with two components T_1, T_2 respectively. If (T_1, T_2) follow the bivariate Gumbel logistic distribution (defined in (4)), then

(i) the failure rate of the parallel system $T_{(2)}$ is given as

$$r_{(2)}(t) = \frac{e^{-t}(e^t + 2)}{2(e^{-t} + 1)^2}, \tag{10}$$

(ii) the failure rate of the series system $T_{(1)}$ is given as

$$r_{(1)}(t) = \frac{e^t(e^t + 2)}{2(e^t + 1)}, \tag{11}$$

Proof. (i) The probability density function of lifetime $T_{(2)}$ is given by

$$f_{(2)}(t) = \frac{2e^t}{(2 + e^t)^2}.$$

The conclusion is reached by recalling equation (7).

(ii) The probability density function of lifetime $T_{(1)}$ can be expressed as

$$f_{(1)}(t) = \frac{2e^t(2e^t + 3)}{(e^{2t} + 3e^t + 2)^2}.$$

The conclusion is reached by recalling equation (8). □

The mean residual lifetime (*MRL*) of a structure is an important characteristic that determines its reliability and quality in time (see, e.g. Eryilmaz, Koutras and Triantafyllou (2011)). The next proposition offers formulae for the computation of the *MRL* function for the reliability systems mentioned in the present section.

Proposition 5. Let $T_{(1)}, T_{(2)}$ denote the lifetimes of a series and a parallel system with two components T_1, T_2 respectively. If (T_1, T_2) follow the bivariate Gumbel logistic distribution (defined in (4)), then

(i) the mean residual lifetime (*MRL*) of the parallel system $T_{(2)}$ is given as

$$m_{(2)}(t) = -\frac{1}{R_{(2)}(t)}\ln(1 - R_{(2)}(t)) \tag{12}$$

(ii) the mean residual lifetime (MRL) of the series system $T_{(1)}$ is given as

$$m_{(1)}(t) = -\frac{1}{R_{(1)}(t)}\ln(1 - R_{(1)}(t)) \ . \tag{13}$$

Proof. (i) By definition, the mean residual lifetime of $T_{(2)}$ can be expressed as

$$m_{(2)}(t) = E(T_{(2)} - t | T_{(2)} > t) = \frac{1}{R_{(2)}(t)}\int_t^\infty R_{(2)}(x)dx, \tag{14}$$

where $R_{(2)}(t)$ is the respective reliability function (see formula (7)). The integral in the above equality, may be rewritten as

$$\int_t^\infty R_{(2)}(x)dx = \int_t^\infty \frac{2}{2+e^x}dx = -\int_{R_{(2)}(t)}^0 (1-u)^{-1}du$$

by employing the transformation $u = 2(2+e^x)^{-1}$. We next replace the last expression in (14) and the proof is complete.

(ii) Employing analogous arguments as in part (i) and using the transformation $u = 2(2+3e^x + e^{2x})^{-1}$ the result is readily deduced. □

Remark 2. Based on the above results, the mean time to failure (*MTTF*) of both series and parallel structures can be easily computed. More specifically, let us denote by $MTTF_{(1)}$ and $MTTF_{(2)}$ the mean time to failure of a series and a parallel structure with two components (T_1, T_2) that follow the bivariate Gumbel logistic distribution. Then, we may deduce that

$$MTTF_{(1)} = E(T_{(1)}) = \int_0^\infty R_{(1)}(x)dx = \ln(4/3),$$

$$MTTF_{(2)} = E(T_{(2)}) = \int_0^\infty R_{(2)}(x)dx = \ln(3).$$

The following figures display the failure rates and the *MRL* functions of a series and a parallel system with two components that follow the bivariate Gumbel logistic distribution.

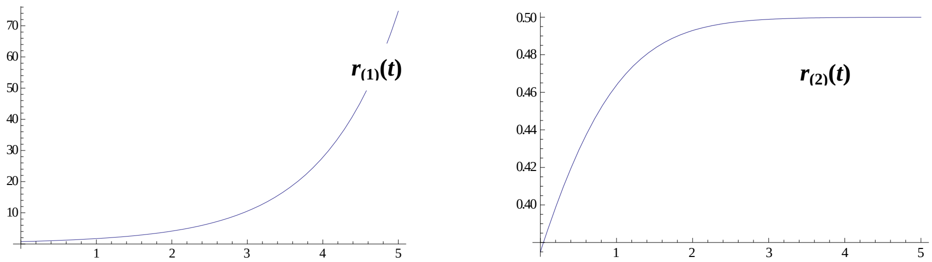

Figure 1: Failure rates of $T_{(1)}$ and $T_{(2)}$ for Gumbel bivariate logistic distribution

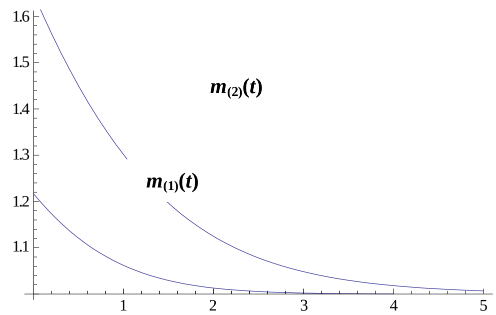

Figure 2: *MRL* functions of $T_{(1)}$ and $T_{(2)}$ for Gumbel bivariate logistic distribution

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